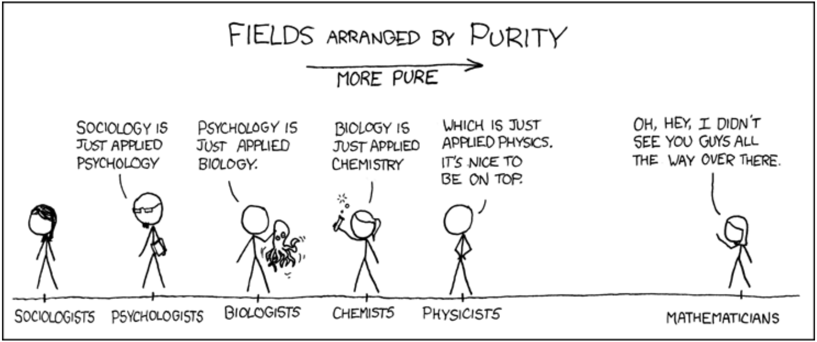


https://xkcd.com/435

- Agenda: an introduction to
  1. **propositional logic**,
  2. **Boolean algebra**, and
  3. application of, i.e., use-cases and rationale for the above within the context of COMS10015.

Part 1: propositional logic (1)

- A **proposition** is basically a statement

the temperature is $20^\circ C$

this statement is false
the temperature is too hot

Part 1: propositional logic (1)

- A **proposition** is basically a statement

  the temperature is $20°C$

  ~~this statement is false~~
  the temperature is too hot

  whose meaning

  1. can be **evaluated** to yield a **truth value**, i.e., **false** or **true**.

Part 1: propositional logic (1)

- A **proposition** is basically a statement

the temperature is $20^{\circ}C$

~~this statement is false~~
~~the temperature is too hot~~

1. can be **evaluated** to yield a **truth value**, i.e., **false** or **true**,
2. must be unambiguous.

Part 1: propositional logic (1)

- A **proposition** is basically a statement

  the temperature is $20°C$
  the temperature is $x°C$
  ~~this statement is false~~
  ~~the temperature is too hot~~

1. can be **evaluated** to yield a **truth value**, i.e., **false** or **true**,
2. must be unambiguous,
3. can include free **variables**.

Part 1: propositional logic (1)

- A **proposition** is basically a statement

$$
\begin{array}{rcl}
f & = & \text{the temperature is } 20^{\circ}C \\
g(x) & = & \text{the temperature is } x^{\circ}C \\
 & & \sout{\text{this statement is false}} \\
 & & \sout{\text{the temperature is too hot}}
\end{array}
$$

1. can be **evaluated** to yield a **truth value**, i.e., **false** or **true**,
2. must be unambiguous,
3. can include free **variables**, and
4. can be represented using a short-hand variable or function, whereby free variables must be bound to concrete arguments before evaluation.

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

  the temperature is not $20°C$

  adding parentheses where needed to add clarity, so that

  1. "not x" is denoted $\neg x$,

- Single statements can be combined using various **connectives**, e.g.,

$$\neg(\text{the temperature is } 20^{\circ}C)$$

  adding parentheses where needed to add clarity, so that
  1. "not x" is denoted $\neg x$,

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

  the temperature is 20°C and it is sunny

  adding parentheses where needed to add clarity, so that
  1. "not x" is denoted $\neg x$,
  2. "x and y" is denoted $x \wedge y$,

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$(\text{the temperature is } 20°C) \wedge (\text{it is sunny})$$

adding parentheses where needed to add clarity, so that

1. "not x" is denoted $\neg x$,
2. "x and y" is denoted $x \wedge y$,

- Single statements can be combined using various **connectives**, e.g.,

  the temperature is $20°C$ or it is sunny

  adding parentheses where needed to add clarity, so that

  1. "not x" is denoted $\neg x$,
  2. "x and y" is denoted $x \wedge y$,
  3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$(\text{the temperature is } 20°C) \vee (\text{it is sunny})$$

  adding parentheses where needed to add clarity, so that

  1. "not x" is denoted $\neg x$,
  2. "x and y" is denoted $x \wedge y$,
  3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

  either the temperature is 20°C or it is sunny, but not both

  adding parentheses where needed to add clarity, so that

  1. "not x" is denoted $\neg x$,
  2. "x and y" is denoted $x \wedge y$,
  3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
  4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,

- Single statements can be combined using various **connectives**, e.g.,

$$\text{(the temperature is } 20^\circ C\text{)} \oplus \text{(it is sunny)}$$

  adding parentheses where needed to add clarity, so that
  1. "not x" is denoted $\neg x$,
  2. "x and y" is denoted $x \wedge y$,
  3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
  4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

  the temperature being 20°C implies that it is sunny

  adding parentheses where needed to add clarity, so that

  1. "not x" is denoted $\neg x$,
  2. "x and y" is denoted $x \wedge y$,
  3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
  4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,
  5. "x implies y" is denoted $x \Rightarrow y$, and sometimes written "if x then y", and

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$(\text{the temperature is } 20°C) \Rightarrow (\text{it is sunny})$$

adding parentheses where needed to add clarity, so that

1. "not x" is denoted $\neg x$,
2. "x and y" is denoted $x \wedge y$,
3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,
5. "x implies y" is denoted $x \Rightarrow y$, and sometimes written "if x then y", and

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

  the temperature is $20^{\circ}C$ is equivalent to it being sunny

  adding parentheses where needed to add clarity, so that

  1. "not x" is denoted $\neg x$,
  2. "x and y" is denoted $x \wedge y$,
  3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
  4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,
  5. "x implies y" is denoted $x \Rightarrow y$, and sometimes written "if x then y", and
  6. "x is equivalent to y" is denoted $x \equiv y$, and sometimes written "x if and only if y" or "x iff. y".

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$(\text{the temperature is } 20^{\circ}C) \equiv (\text{it is sunny})$$

adding parentheses where needed to add clarity, so that

1. "not x" is denoted $\neg x$,
2. "x and y" is denoted $x \wedge y$,
3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,
5. "x implies y" is denoted $x \Rightarrow y$, and sometimes written "if x then y", and
6. "x is equivalent to y" is denoted $x \equiv y$, and sometimes written "x if and only if y" or "x iff. y".

Part 1: propositional logic (2)

- You *might* see more formal terms or different notation for the *same* connectives:
  - $\neg$ is often termed logical **compliment** (or **negation**),
  - $\wedge$ is often termed logical **conjunction**,
  - $\vee$ is often termed logical (inclusive) **disjunction**,
  - $\oplus$ is often termed logical (exclusive) **disjunction**,
  - $\Rightarrow$ is often termed logical **implication**, and
  - $\equiv$ is often termed logical **equivalence**.

- You can think of the same thing diagrammatically, i.e.,

$$r \;=\; (\text{the temperature is } 20^{\circ}C) \wedge (\text{it is sunny})$$

$$\equiv$$

but either way, the question is how do we **evaluate** the (compound) proposition (or **expression**) to produce a truth value?

- Since each statement can only evaluate to **true** or **false**, we can enumerate all possible outcomes in a **truth table**, e.g., if

$$\begin{array}{rcl} x & = & \text{the temperature is } 20^\circ C \\ y & = & \text{it is sunny} \\ r & = & (\text{the temperature is } 20^\circ\text{C}) \wedge (\text{it is sunny}) \end{array}$$

  then

  (inputs: x, y; output: r)

x	y	r
false	**false**	**false**
false	**true**	**false**
true	**false**	**false**
true	**true**	**true**

- Note that
  1. each row details the output(s) associated with a given assignment to the inputs,
  2. if there are n inputs, the truth table will have 2^n rows.

Part 1: propositional logic (5)

Definition

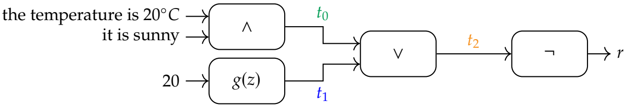
x	y	$\neg x$	$x \wedge y$	$x \vee y$	$x \oplus y$	$x \Rightarrow y$	$x \equiv y$
false	**false**	**true**	**false**	**false**	**false**	**true**	**true**
false	**true**	**true**	**false**	**true**	**true**	**true**	**false**
true	**false**	**false**	**false**	**true**	**true**	**false**	**false**
true	**true**	**false**	**true**	**true**	**false**	**true**	**true**

Example

Imagine that now

$$
\begin{array}{lcl}
x & = & \text{the temperature is } 20^\circ C \\
y & = & \text{it is sunny} \\
g(z) & = & \text{the temperature is } z^\circ C \\
r & = & \neg(((\text{the temperature is } 20^\circ C) \wedge (\text{it is sunny})) \vee (\text{the temperature is } z^\circ C))
\end{array}
$$

which we translate into the diagrammatic form



An example evaluation might be as follows:

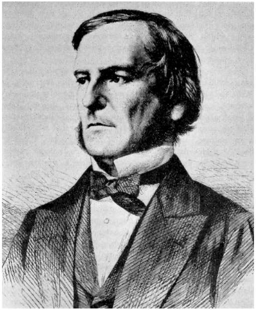
inputs		intermediates			output
x	y	t_0	t_1	t_2	r
false	**false**	**false**	**false**	**false**	**true**
false	**true**	**false**	**false**	**false**	**true**
true	**false**	**false**	**true**	**true**	**false**
true	**true**	**true**	**true**	**true**	**false**

- Notice that
  1. in **elementary algebra**, for some number x we have that
     $$x + 0 = x$$
     and
     $$x \cdot 1 = x,$$
  2. in **set theory**, for some set x we have that
     $$x \cup \emptyset = x$$
     and
     $$x \cap \mathcal{U} = x,$$
     plus we've now demonstrated that
  3. in **propositional logic**, for some truth value x we have that
     $$x \vee \textbf{false} = x$$
     and
     $$x \wedge \textbf{true} = x.$$

Thou must

1. work with the set $\mathbb{B} = \{0, 1\}$ of **binary** digits, using 0 and 1 instead of **false** and **true**,
2. shorten every statement into either a **variable** *or* **function**,
3. use **unary operators**, e.g., $\neg$ (or NOT), and **binary operators**, e.g., $\wedge$ and $\vee$ (or AND and OR), to form **expressions**,
4. manipulate said expressions according to some axioms (or rules),

then call the result **Boolean algebra**.



https://en.wikipedia.org/wiki/File:George_Boole.jpg

Part 2: Boolean algebra (3)

- Put more concretely, we now have
  1. a set of operators specified by

Definition

x	y	$\neg x$	$x \wedge y$	$x \vee y$	$x \oplus y$	$x \Rightarrow y$	$x \equiv y$
0	0	1	0	0	0	1	1
0	1	1	0	1	1	1	0
1	0	0	0	1	1	0	0
1	1	0	1	1	0	1	1

Part 2: Boolean algebra (3)

- Put more concretely, we now have

2. a set of axioms that allow manipulation of expressions comprised of said operators, i.e.,

Definition

Name	Axiom(s)	Name	Axiom(s)
commutativity	$x \wedge y \equiv y \wedge x$	commutativity	$x \vee y \equiv y \vee x$
association	$(x \wedge y) \wedge z \equiv x \wedge (y \wedge z)$	association	$(x \vee y) \vee z \equiv x \vee (y \vee z)$
distribution	$x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$	distribution	$x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$

plus other rules such as **precedence** (to deal with ambiguity in the absence of parentheses).

- Put more concretely, we now have

2. a set of axioms that allow manipulation of expressions comprised of said operators, i.e.,

Definition

Name	Axiom(s)	Name	Axiom(s)
identity	$x \wedge 1 \equiv x$	identity	$x \vee 0 \equiv x$
null	$x \wedge 0 \equiv 0$	null	$x \vee 1 \equiv 1$
idempotency	$x \wedge x \equiv x$	idempotency	$x \vee x \equiv x$
inverse	$x \wedge \neg x \equiv 0$	inverse	$x \vee \neg x \equiv 1$

plus other rules such as **precedence** (to deal with ambiguity in the absence of parentheses).

- Put more concretely, we now have

2. a set of axioms that allow manipulation of expressions comprised of said operators, i.e.,

Definition

Name	Axiom(s)	Name	Axiom(s)
absorption	$x \wedge (x \vee y) \equiv x$	absorption	$x \vee (x \wedge y) \equiv x$
de Morgan	$\neg(x \wedge y) \equiv \neg x \vee \neg y$	de Morgan	$\neg(x \vee y) \equiv \neg x \wedge \neg y$

plus other rules such as **precedence** (to deal with ambiguity in the absence of parentheses).

- Put more concretely, we now have

2. a set of axioms that allow manipulation of expressions comprised of said operators, i.e.,

Definition

Name	Axiom(s)
equivalence	$x \equiv y \;\equiv\; (x \Rightarrow y) \wedge (y \Rightarrow x)$
implication	$x \Rightarrow y \;\equiv\; \neg x \vee y$
involution	$\neg\neg x \;\equiv\; x$

plus other rules such as **precedence** (to deal with ambiguity in the absence of parentheses).

Definition

Consider a Boolean expression:

1. When the expression is written as a sum (i.e., OR) of terms which each comprise the product (i.e., AND) of variables, e.g.,

$$\underbrace{(a \wedge b \wedge c)}_{\text{minterm}} \vee (d \wedge e \wedge f),$$

   it is said to be in **disjunctive normal form** or **Sum of Products (SoP)** form; the terms are called the **minterms**. Note that each variable can exist as-is *or* complemented using NOT, meaning

$$\underbrace{(\neg a \wedge b \wedge c)}_{\text{minterm}} \vee (d \wedge \neg e \wedge f),$$

   is also a valid SoP expression.

2. When the expression is written as a product (i.e., AND) of terms which each comprise the sum (i.e., OR) of variables, e.g.,

$$\underbrace{(a \vee b \vee c)}_{\text{maxterm}} \wedge (d \vee e \vee f),$$

   it is said to be in **conjunctive normal form** or **Product of Sums (PoS)** form; the terms are called the **maxterms**. As above each variable can exist as-is *or* complemented using NOT.

- Concept: we can define various **derived operators** in terms of NOT, AND, and OR.
- Example:
  - "exclusive-OR" or **XOR**, such that

$$x \oplus y \equiv (\neg x \wedge y) \vee (x \wedge \neg y)$$

so

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

- Concept: we can define various **derived operators** in terms of NOT, AND, and OR.
- Example:
  - "NOT-AND" or **NAND**, such that

$$x \overline{\wedge} y \equiv \neg(x \wedge y)$$

so

x	y	$x \overline{\wedge} y$
0	0	1
0	1	1
1	0	1
1	1	0

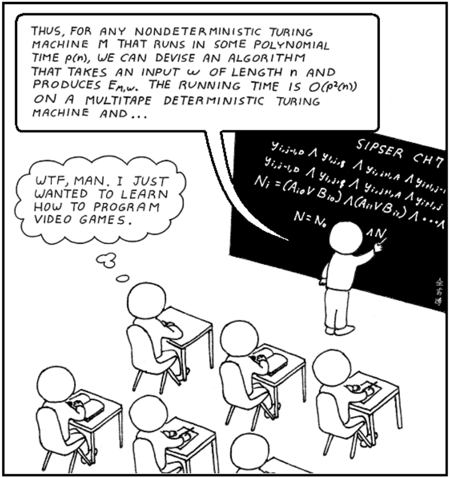
  - "NOT-OR" or **NOR**, such that

$$x \overline{\vee} y \equiv \neg(x \vee y)$$

so

x	y	$x \overline{\vee} y$
0	0	1
0	1	0
1	0	0
1	1	0

Part 3: application (1)



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https://abstrusegoose.com/206

Part 3: application (2)

- (Fairly) reasonable question(s):
  1. "I thought this was CS, not Maths!", and
  2. "why does *this* unit duplicate material in *other* units?".

Part 3: application (2)

- (Fairly) reasonable question(s):
  1. "I thought this was CS, not Maths!", and
  2. "why does *this* unit duplicate material in *other* units?".
- Answer: it isn't, and it doesn't (well, not *too* much) ... note that
  - theoretical concepts, e.g., often have significant practical motivations or implications, and
  - it's perfectly reasonable to utilise **Electronic Design Automation (EDA)** [3] tools.

- Question: simplify the Boolean expression

$$(\neg(a \vee b) \wedge \neg(c \vee d \vee e)) \vee \neg(a \vee b)$$

into a form that contains the fewest operators possible.

- Question: simplify the Boolean expression

$$(\neg(a \vee b) \wedge \neg(c \vee d \vee e)) \vee \neg(a \vee b)$$

  into a form that contains the fewest operators possible.

- Solution #1: less steps.

$$\begin{array}{lllllll} & (\neg(a \vee b) & \wedge & \neg(c \vee d \vee e)) & \vee & \neg(a \vee b) & \\ = & \neg(a \vee b) & \vee & (\neg(a \vee b) & \wedge & \neg(c \vee d \vee e)) & \text{(commutativity)} \\ = & \neg(a \vee b) & & & & & \text{(absorption)} \end{array}$$

- Question: simplify the Boolean expression

$$(\neg(a \vee b) \wedge \neg(c \vee d \vee e)) \vee \neg(a \vee b)$$

  into a form that contains the fewest operators possible.
- Solution #2: more steps.

$$
\begin{array}{lllllll}
 & (\neg(a \vee b) & \wedge & \neg(c \vee d \vee e)) & \vee & \neg(a \vee b) & \\
= & ((\neg a \wedge \neg b) & \wedge & \neg(c \vee d \vee e)) & \vee & \neg(a \vee b) & \text{(de Morgan)} \\
= & ((\neg a \wedge \neg b) & \wedge & (\neg c \wedge \neg d \wedge \neg e)) & \vee & \neg(a \vee b) & \text{(de Morgan)} \\
= & ((\neg a \wedge \neg b) & \wedge & (\neg c \wedge \neg d \wedge \neg e)) & \vee & (\neg a \wedge \neg b) & \text{(de Morgan)} \\
= & (\neg a \wedge \neg b) & \vee & ((\neg a \wedge \neg b) & \wedge & (\neg c \wedge \neg d \wedge \neg e)) & \text{(commutativity)} \\
= & (\neg a \wedge \neg b) & & & & & \text{(absorption)} \\
= & \neg(a \vee b) & & & & & \text{(de Morgan)}
\end{array}
$$

- **Question**: simplify the Boolean expression

$$(a \wedge b \wedge c) \vee (\neg a \wedge b) \vee (a \wedge b \wedge \neg c)$$

into a form that contains the fewest operators possible.

- **Question**: simplify the Boolean expression

$$(a \wedge b \wedge c) \vee (\neg a \wedge b) \vee (a \wedge b \wedge \neg c)$$

into a form that contains the fewest operators possible.

- **Solution**:

$$
\begin{array}{llllllr}
 & (a \wedge b \wedge c) & \vee & (\neg a \wedge b) & \vee & (a \wedge b \wedge \neg c) & \\
= & (a \wedge b \wedge c) & \vee & (a \wedge b \wedge \neg c) & \vee & (\neg a \wedge b) & \text{(commutativity)} \\
= & (a \wedge b) & \wedge & (c \vee \neg c) & \vee & (\neg a \wedge b) & \text{(distribution)} \\
= & (a \wedge b) & \wedge & 1 & \vee & (\neg a \wedge b) & \text{(inverse)} \\
= & (a \wedge b) & & & \vee & (\neg a \wedge b) & \text{(identity)} \\
= & b & \wedge & (a \vee \neg a) & & & \text{(distribution)} \\
= & b & \wedge & 1 & & & \text{(inverse)} \\
= & b & & & & & \text{(identity)}
\end{array}
$$

Quote

If I designed a computer with 200 chips, I tried to design it with 150. And then I would try to design it with 100. I just tried to find every trick I could in life to design things real tiny.

– Wozniak

Quote

So I took 20 chips off their board; I bypassed 20 of their chips.

– Wozniak

Part 3: application (6)
Axiomatic manipulation ⇝ optimisation

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https://en.wikipedia.org/wiki/File:Shugart_SA400.jpg

https://en.wikipedia.org/wiki/File:Interface_Card_-_Disk_II_Interface_Apple2.jpg

- Concept: truth tables can accommodate **don't care** entries, e.g.,

x	y	r
?	0	1
0	1	**?**
1	1	0

such that

- a **?** (rather than 0 or 1) means we "don't care" (≠ "don't know"),
- on the LHS, for an *in*put,
  - **?** is a wildcard (or short-hand),
  - it means 0 *and* 1,
  - we've compressed two truth table rows into one.
- on the RHS, for an *out*put,
  - **?** is a choice,
  - it means 0 *or* 1,
  - we can select which one to, e.g., optimise the associated expression.

- **Fact**: NAND and NOR are **functionally complete** (or **universal**), e.g.,

$$\begin{array}{lcccl} \neg x & & & \equiv & x \overline{\wedge} x \\ x \wedge y & & & \equiv & (x \overline{\wedge} y) \overline{\wedge} (x \overline{\wedge} y) \\ x \vee y & \equiv & \neg x \overline{\wedge} \neg y & \equiv & (x \overline{\wedge} x) \overline{\wedge} (y \overline{\wedge} y) \end{array}$$

which we can prove via

x	y	$x \overline{\wedge} y$	$x \overline{\wedge} x$	$y \overline{\wedge} y$	$(x \overline{\wedge} y) \overline{\wedge} (x \overline{\wedge} y)$	$(x \overline{\wedge} x) \overline{\wedge} (y \overline{\wedge} y)$
0	0	1	1	1	0	0
0	1	1	1	0	0	1
1	0	1	0	1	0	1
1	1	0	0	0	1	1

$\therefore$ *any* Boolean function can be expressed using a *single* operator.

- **Question**: translate

$$x \wedge (y \vee z)$$

into a version using NAND only.

- Question: translate

$$x \wedge (y \vee z)$$

into a version using NAND only.

- Solution #1: apply the identities *naively* to get

$$\begin{array}{cl} & x \wedge (y \vee z) \\ = & x \wedge ((y \overline{\wedge} y) \overline{\wedge} (z \overline{\wedge} z)) \\ = & (x \overline{\wedge} ((y \overline{\wedge} y) \overline{\wedge} (z \overline{\wedge} z))) \overline{\wedge} (x \overline{\wedge} ((y \overline{\wedge} y) \overline{\wedge} (z \overline{\wedge} z))) \end{array}$$

- Question: translate

$$x \wedge (y \vee z)$$

into a version using NAND only.

- Solution #2: apply the identities *intelligently* to get

$$\begin{array}{rl} & x \wedge (y \vee z) \\ = & x \wedge ((y \overline{\wedge} y) \overline{\wedge} (z \overline{\wedge} z)) \\ = & t \overline{\wedge} t \end{array}$$

where $t = x \overline{\wedge} ((y \overline{\wedge} y) \overline{\wedge} (z \overline{\wedge} z))$ is a common sub-expression [2].

Conclusions

- Take away points:
  1. The design of computational devices, e.g., micro-processors, *isn't* ad hoc: Boolean algebra offers a theoretical basis for reasoning about computational devices (and computation) in practice.

Conclusions

- Take away points:

2. Boolean algebra is a (somewhat) cosmetic extension of what you already know.
3. Keep in mind that
   - *any* Boolean function f which can be expressed by a truth table can be computed using an associated Boolean expression,
   - a Boolean expression is composed of Boolean operators,
   - *if* we (physically) implement the Boolean operators, we can implement the Boolean expression and hence compute f.

Conclusions

- Take away points:

4. We'll focus on *application* (i.e., *use*) vs. *theory* (e.g., *study*) of Boolean algebra from here on.
5. Keep in mind that
   - "it works" $\neq$ "it works *well*",
   - using automation is fine *iff.* you know the underlying theory,
   - using brute-force is fine *iff.* you know the underlying theory,
   - Boolean algebra > Boolean axioms: concepts that *seem* of interest in theory alone, *can* be important if/when applied in practice.

Additional Reading

- *Wikipedia: Boolean algebra*. URL: https://en.wikipedia.org/wiki/Boolean_algebra.
- D. Page. "Chapter 1: Mathematical preliminaries". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009.
- W. Stallings. "Chapter 11: Digital logic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013.
- A.S. Tanenbaum and T. Austin. "Section 3.1: Gates and Boolean algebra". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012.

References

[1] *Wikipedia: Boolean algebra*. URL: https://en.wikipedia.org/wiki/Boolean_algebra (see p. 53).

[2] *Wikipedia: Common sub-expression elimination*. URL: https://en.wikipedia.org/wiki/Common_subexpression_elimination (see pp. 48, 49).

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[4] D. Page. "Chapter 1: Mathematical preliminaries". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009 (see p. 53).

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