- **Concept**: consider

$$\begin{array}{ccc} \hat{x} & \mapsto & x \\ \hat{y} & \mapsto & y \end{array}$$

- **Concept**: consider

$$\begin{array}{rcl} \hat{x} & \mapsto & x \\ \hat{y} & \mapsto & y \\ & & r = x + y \end{array}$$

- **Concept**: consider

$$\begin{array}{rcccccc} & & \hat{x} & \mapsto & x & & \\ & & \hat{y} & \mapsto & y & & \\ f(\hat{x},\hat{y}) & = & \hat{r} & \mapsto & r & = & x+y \end{array}$$

where f

1. has an action on $\hat{x}$ and $\hat{y}$ compatible with that of $+$ on x and y:
   - accepts n-bit
     - **addend** $\hat{x}$, and
     - **addend** $\hat{y}$

     as input, and
   - produces an $(n+1)$-bit **sum** $\hat{r}$ as output,
2. is a Boolean function:

$$f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^{n+1}$$

- Agenda: produce a design(s) for f, which
  1. functions correctly, and
  2. satisfies pertinent quality metrics (e.g., is efficient in time and/or space).

Part 1: addition in theory (1)

- Concept:

Example ($b = 10$)

$$
\begin{array}{lclcccccc}
x & = & 107_{(10)} & \mapsto & & 1 & 0 & 7 & \\
y & = & 14_{(10)} & \mapsto & & 0 & 1 & 4 & + \\
\hline
c & = & & & & & & & \\
r & = & & & & & & & \\
\hline
\end{array}
$$

Example ($b = 2$)

$$
\begin{array}{lclcccccccccc}
x & = & 107_{(10)} & \mapsto & & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
y & = & 14_{(10)} & \mapsto & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \; + \\
\hline
c & = & & & & & & & & & & & \\
r & = & & & & & & & & & & & \\
\hline
\end{array}
$$

$\therefore$

1. this process matches our understanding of manual, “school-book” addition, *and*
2. the same process applies, irrespective of b.

Part 1: addition in theory (1)

- Concept:

Example ($b = 10$)

$$
\begin{array}{lclcrrrl}
x & = & 107_{(10)} & \mapsto & 1 & 0 & 7 & \\
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c & = & & & & & 0 & \\
r & = & & & & & &
\end{array}
$$

Example ($b = 2$)

$$
\begin{array}{lclcrrrrrrrrl}
x & = & 107_{(10)} & \mapsto & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & \\
y & = & 14_{(10)} & \mapsto & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & + \\
c & = & & & & & & & & & & & \\
r & = & & & & & & & & & & &
\end{array}
$$

$\therefore$

1. this process matches our understanding of manual, "school-book" addition, *and*
2. the same process applies, irrespective of b.

Part 1: addition in theory (1)

- Concept:

Example ($b = 10$)

x	=	$107_{(10)}$	$\mapsto$	1	0	7	
y	=	$14_{(10)}$	$\mapsto$	0	1	4	+
c	=				1	0	
r	=					1	

Example ($b = 2$)

x	=	$107_{(10)}$	$\mapsto$	0	1	1	0	1	0	1	1	
y	=	$14_{(10)}$	$\mapsto$	0	0	0	0	1	1	1	0	+
c	=											
r	=											

$\therefore$

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Part 1: addition in theory (1)

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\begin{array}{lclcccccc}
x & = & 107_{(10)} & \mapsto & & 1 & 0 & 7 & \\
y & = & 14_{(10)} & \mapsto & & 0 & 1 & 4 & + \\
\hline
c & = & & & & 0 & 1 & 0 & \\
r & = & & & & & 2 & 1 & \\
\hline
\end{array}
$$

Example ($b = 2$)

$$
\begin{array}{lclcccccccccc}
x & = & 107_{(10)} & \mapsto & & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & \\
y & = & 14_{(10)} & \mapsto & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & + \\
\hline
c & = & & & & & & & & & & & & \\
r & = & & & & & & & & & & & & \\
\hline
\end{array}
$$

$\therefore$

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Part 1: addition in theory (1)

- Concept:

Example ($b = 10$)

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\begin{array}{lclcrrrrl}
x & = & 107_{(10)} & \mapsto & & 1 & 0 & 7 & \\
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\hline
c & = & & & 0 & 0 & 1 & 0 & \\
r & = & 121_{(10)} & \mapsto & & 1 & 2 & 1 & \\
\hline
\end{array}
$$

Example ($b = 2$)

$$
\begin{array}{lclcrrrrrrrrrl}
x & = & 107_{(10)} & \mapsto & & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & \\
y & = & 14_{(10)} & \mapsto & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & + \\
\hline
c & = & & & & & & & & & & & & \\
r & = & & & & & & & & & & & & \\
\hline
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Part 1: addition in theory (1)

- Concept:

Example ($b = 10$)

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Part 1: addition in theory (1)

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r & = & 121_{(10)} & \mapsto & & 1 & 2 & 1 &
\end{array}
$$

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\begin{array}{rclccccccccl}
x & = & 107_{(10)} & \mapsto & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & \\
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c & = & & & & & & & & & 0 & 0 & \\
r & = & & & & & & & & & & 1 &
\end{array}
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Part 1: addition in theory (1)

- Concept:

Example ($b = 10$)

$$
\begin{array}{rcrcl}
x & = & 107_{(10)} & \mapsto & \begin{array}{cccc} & 1 & 0 & 7 \end{array} \\
y & = & 14_{(10)} & \mapsto & \begin{array}{cccc} & 0 & 1 & 4 \end{array} + \\
c & = & & & \begin{array}{cccc} 0 & 0 & 1 & 0 \end{array} \\
r & = & 121_{(10)} & \mapsto & \begin{array}{cccc} & 1 & 2 & 1 \end{array}
\end{array}
$$

Example ($b = 2$)

$$
\begin{array}{rcrcl}
x & = & 107_{(10)} & \mapsto & \begin{array}{cccccccc} 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \end{array} \\
y & = & 14_{(10)} & \mapsto & \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array} + \\
c & = & & & \begin{array}{cccccccc} & & & & & 1 & 0 & 0 \end{array} \\
r & = & & & \begin{array}{cccccccc} & & & & & & 0 & 1 \end{array}
\end{array}
$$

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1. this process matches our understanding of manual, "school-book" addition, *and*
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Part 1: addition in theory (1)

- **Concept**:

Example ($b = 10$)

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\begin{array}{lcrcccccl}
x & = & 107_{(10)} & \mapsto & & 1 & 0 & 7 & \\
y & = & 14_{(10)} & \mapsto & & 0 & 1 & 4 & + \\
\hline
c & = & & & 0 & 0 & 1 & 0 & \\
r & = & 121_{(10)} & \mapsto & & 1 & 2 & 1 & \\
\hline
\end{array}
$$

Example ($b = 2$)

$$
\begin{array}{lcrcccccccccl}
x & = & 107_{(10)} & \mapsto & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & \\
y & = & 14_{(10)} & \mapsto & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & + \\
\hline
c & = & & & & & & & 1 & 1 & 0 & 0 & \\
r & = & & & & & & & & 0 & 0 & 1 & \\
\hline
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$\therefore$

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Part 1: addition in theory (1)

- Concept:

Example ($b = 10$)

$$
\begin{array}{lcrcrrrrl}
x & = & 107_{(10)} & \mapsto & & 1 & 0 & 7 & \\
y & = & 14_{(10)} & \mapsto & & 0 & 1 & 4 & + \\
\hline
c & = & & & 0 & 0 & 1 & 0 & \\
r & = & 121_{(10)} & \mapsto & & 1 & 2 & 1 & \\
\hline
\end{array}
$$

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\begin{array}{lcrcrrrrrrrrl}
x & = & 107_{(10)} & \mapsto & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & \\
y & = & 14_{(10)} & \mapsto & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & + \\
\hline
c & = & & & & & & 1 & 1 & 1 & 0 & 0 & \\
r & = & & & & & & & 1 & 0 & 0 & 1 & \\
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Part 1: addition in theory (1)

- Concept:

Example ($b = 10$)

$$\begin{array}{lcrcccccc} x & = & 107_{(10)} & \mapsto & & 1 & 0 & 7 & \\ y & = & 14_{(10)} & \mapsto & & 0 & 1 & 4 & + \\ c & = & & & 0 & 0 & 1 & 0 & \\ r & = & 121_{(10)} & \mapsto & & 1 & 2 & 1 & \end{array}$$

Example ($b = 2$)

$$\begin{array}{lcrcccccccccc} x & = & 107_{(10)} & \mapsto & & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & \\ y & = & 14_{(10)} & \mapsto & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & + \\ c & = & & & & & & 0 & 1 & 1 & 1 & 0 & 0 & \\ r & = & & & & & & & 1 & 1 & 0 & 0 & 1 & \end{array}$$

$\therefore$

1. this process matches our understanding of manual, "school-book" addition, *and*
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Part 1: addition in theory (1)

- Concept:

Example ($b = 10$)

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\begin{array}{lclcrrrrl}
x & = & 107_{(10)} & \mapsto & & 1 & 0 & 7 & \\
y & = & 14_{(10)} & \mapsto & & 0 & 1 & 4 & + \\
c & = & & & 0 & 0 & 1 & 0 & \\
r & = & 121_{(10)} & \mapsto & & 1 & 2 & 1 &
\end{array}
$$

Example ($b = 2$)

$$
\begin{array}{lclcrrrrrrrrrl}
x & = & 107_{(10)} & \mapsto & & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & \\
y & = & 14_{(10)} & \mapsto & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & + \\
c & = & & & & & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \\
r & = & & & & & & 1 & 1 & 1 & 0 & 0 & 1 &
\end{array}
$$

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1. this process matches our understanding of manual, “school-book” addition, *and*
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Part 1: addition in theory (1)

- Concept:

Example ($b = 10$)

$$
\begin{array}{lclcccccc}
x & = & 107_{(10)} & \mapsto & & 1 & 0 & 7 & \\
y & = & 14_{(10)} & \mapsto & & 0 & 1 & 4 & + \\
\hline
c & = & & & 0 & 0 & 1 & 0 & \\
r & = & 121_{(10)} & \mapsto & & 1 & 2 & 1 & \\
\hline
\end{array}
$$

Example ($b = 2$)

$$
\begin{array}{lclccccccccc}
x & = & 107_{(10)} & \mapsto & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & \\
y & = & 14_{(10)} & \mapsto & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & + \\
\hline
c & = & & & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \\
r & = & & & & 1 & 1 & 1 & 1 & 0 & 0 & 1 & \\
\hline
\end{array}
$$

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1. this process matches our understanding of manual, "school-book" addition, *and*
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Part 1: addition in theory (1)

- **Concept**:

Example ($b = 10$)

$$
\begin{array}{lclcccccl}
x & = & 107_{(10)} & \mapsto & & 1 & 0 & 7 & \\
y & = & 14_{(10)} & \mapsto & & 0 & 1 & 4 & + \\
\hline
c & = & & & 0 & 0 & 1 & 0 & \\
r & = & 121_{(10)} & \mapsto & & 1 & 2 & 1 & \\
\hline
\end{array}
$$

Example ($b = 2$)

$$
\begin{array}{lclcccccccccl}
x & = & 107_{(10)} & \mapsto & & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
y & = & 14_{(10)} & \mapsto & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \; + \\
\hline
c & = & & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
r & = & 121_{(10)} & \mapsto & & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
\hline
\end{array}
$$

$\therefore$

1. this process matches our understanding of manual, “school-book” addition, *and*
2. the same process applies, irrespective of b.

Algorithm

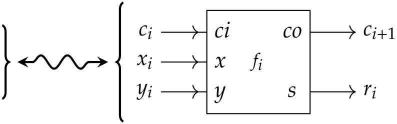
Input: Two unsigned, n-digit, base-b integers x and y, and a 1-digit carry-in $ci \in \{0, 1\}$
Output: An unsigned, n-digit, base-b integer $r = x + y$, and a 1-digit carry-out $co \in \{0, 1\}$

$r \leftarrow 0$, $c_0 \leftarrow ci$
for $i = 0$ **upto** $n - 1$ **step** $+1$ **do**
    $r_i \leftarrow (x_i + y_i + c_i) \bmod b$
    if $(x_i + y_i + c_i) < b$ **then** $c_{i+1} \leftarrow 0$ **else** $c_{i+1} \leftarrow 1$
end
$co \leftarrow c_n$
return r, co

Part 3: addition in practice: a circuit (1)

- **Idea**:
  1. for $b = 2$, it's clear from the algorithm that

     $r_i \leftarrow (x_i + y_i + c_i) \bmod b$
     if $(x_i + y_i + c_i) < b$ **then** $c_{i+1} \leftarrow 0$ **else** $c_{i+1} \leftarrow 1$

  2. the loop body is therefore analagous to a Boolean function

     $$f_i : \{0,1\}^3 \rightarrow \{0,1\}^2$$

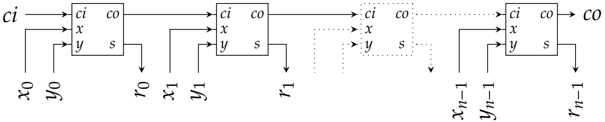
     specified by the following truth table

ci	x	y	co	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Part 3: addition in practice: a circuit (1)

- **Idea**:
  3. the loop bound is fixed, i.e., n is some known constant, so we can unroll it to yield

Circuit



which, now read left-to-right, mirrors the algorithm:

- the i-th instance f_i implements the i-th loop iteration,
- the connection, or **carry chain** between instances captures c,
- those instances are termed **full adder** cells,
- this combination of them is termed a **ripple-carry adder**.

Part 3: addition in practice: a circuit (2)

- Beware:
  - the magnitude of $r = x + y$ can exceed what we can represent via $\hat{r}$:

$\hat{x}$ and $\hat{y}$ are *unsigned*, and there is a carry-out	$\Rightarrow$	**carry** condition
$\hat{x}$ and $\hat{y}$ are *signed*, and the sign of $\hat{r}$ is incorrect	$\Rightarrow$	**overflow** condition

  - to cope, we typically
    1. detect the condition,
    2. potentially take some action (e.g., try to “fix” the result somehow),
    3. potentially signal the condition somehow (e.g., via a status register or some form of exception).

Part 3: addition in practice: a circuit (3)

Example

Consider use of an *unsigned* representation:

$$
\begin{array}{lclcccccccl}
x & = & 15_{(10)} & \mapsto & & 1 & 1 & 1 & 1 & \\
y & = & 1_{(10)} & \mapsto & & 0 & 0 & 0 & 1 & + \\
\hline
c & = & & & 1 & 1 & 1 & 1 & 0 & \\
r & = & 0_{(10)} & \mapsto & & 0 & 0 & 0 & 0 & \\
\hline
\end{array}
$$

Here, the carry-out indicates an error: the correct result $r = 16$ is too large for $n = 4$ bits.

- Note that
  1. detection:

     $$
     \begin{array}{lcr}
     c_n = co = 0 & \Rightarrow & \text{no carry} \\
     c_n = co = 1 & \Rightarrow & \text{carry}
     \end{array}
     $$

  2. action, e.g., **truncate** the result to n bits.

Part 3: addition in practice: a circuit (4)

Example

Consider use of a *signed* representation:

$$
\begin{array}{lclcrrrrrl}
x & = & -1_{(10)} & \mapsto & & 1 & 1 & 1 & 1 & \\
y & = & 1_{(10)} & \mapsto & & 0 & 0 & 0 & 1 & + \\
\hline
c & = & & & 1 & 1 & 1 & 1 & 0 & \\
r & = & 0_{(10)} & \mapsto & & 0 & 0 & 0 & 0 & \\
\end{array}
$$

Irrespective of the carry-out, the signs of inputs and output make sense: there is no overflow, so $r = 0$ is correct.

Example

Consider use of a *signed* representation:

$$
\begin{array}{lclcrrrrrl}
x & = & 7_{(10)} & \mapsto & & 0 & 1 & 1 & 1 & \\
y & = & 1_{(10)} & \mapsto & & 0 & 0 & 0 & 1 & + \\
\hline
c & = & & & 0 & 1 & 1 & 1 & 0 & \\
r & = & -8_{(10)} & \mapsto & & 1 & 0 & 0 & 0 & \\
\end{array}
$$

Irrespective of the carry-out, the signs of inputs and output make no sense: there is an overflow, so $r = -8$ is incorrect.

- Note that
  1. detection:

$$
\begin{array}{llll}
x \text{ +ve} & y \text{ -ve} & & \Rightarrow \text{ no overflow} \\
x \text{ -ve} & y \text{ +ve} & & \Rightarrow \text{ no overflow} \\
x \text{ +ve} & y \text{ +ve} & r \text{ +ve} & \Rightarrow \text{ no overflow} \\
x \text{ +ve} & y \text{ +ve} & r \text{ -ve} & \Rightarrow \text{ overflow} \\
x \text{ -ve} & y \text{ -ve} & r \text{ +ve} & \Rightarrow \text{ overflow} \\
x \text{ -ve} & y \text{ -ve} & r \text{ -ve} & \Rightarrow \text{ no overflow} \\
\end{array}
$$

  2. action, e.g., **clamp** (or **saturate**) the result to the largest magnitude representable in n bits.

Conclusions

- Take away points:
  1. Computer arithmetic is a broad, interesting (sub-)field:
     - it's a broad topic with a rich history,
     - there's usually a large design space of potential approaches,
     - they're often easy to understand at an intuitive, high level,
     - correctness and efficiency of resulting low-level solutions is vital and challenging.
  2. The strategy we've employed is important and (fairly) general-purpose:
     - explore and understand an approach in theory,
     - translate, formalise, and generalise the approach into an algorithm,
     - translate the algorithm, e.g., into circuit,
     - refine (or select) the circuit to satisfy any design constraints.

Additional Reading

- *Wikipedia: Computer Arithmetic*. URL: https://en.wikipedia.org/wiki/Category:Computer_arithmetic.
- D. Page. "Chapter 7: Arithmetic and logic". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009.
- B. Parhami. "Part 2: Addition/subtraction". In: *Computer Arithmetic: Algorithms and Hardware Designs*. 1st ed. Oxford University Press, 2000.
- W. Stallings. "Chapter 10: Computer arithmetic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013.
- A.S. Tanenbaum and T. Austin. "Section 3.2.2: Arithmetic circuits". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012.

References

[1] *Wikipedia: Computer Arithmetic*. URL: https://en.wikipedia.org/wiki/Category:Computer_arithmetic (see p. 26).

[2] D. Page. "Chapter 7: Arithmetic and logic". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009 (see p. 26).

[3] B. Parhami. "Part 2: Addition/subtraction". In: *Computer Arithmetic: Algorithms and Hardware Designs*. 1st ed. Oxford University Press, 2000 (see p. 26).

[4] W. Stallings. "Chapter 10: Computer arithmetic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013 (see p. 26).

[5] A.S. Tanenbaum and T. Austin. "Section 3.2.2: Arithmetic circuits". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012 (see p. 26).