

# Computer Architecture

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Keep in mind there are *two* PDFs available (of which this is the latter):

1. a PDF of examinable material used as lecture slides, and
2. a PDF of non-examinable, extra material:
  - ▶ the associated notes page may be pre-populated with extra, written explanation of material covered in lecture(s), plus
  - ▶ anything with a “grey’ed out” header/footer represents extra material which is useful and/or interesting but out of scope (and hence not covered).

Notes:

Notes:

- **Problem:** we have  $\neg$ ,  $\wedge$ , and  $\vee$ , so given the specification

$x_{n-1}$	$\cdots$	$x_1$	$x_0$	$r$
0	$\cdots$	0	0	1
0	$\cdots$	0	1	0
0	$\cdots$	1	0	1
0	$\cdots$	1	1	0
$\vdots$		$\vdots$	$\vdots$	$\vdots$
1	$\cdots$	1	1	0

for some Boolean function

$$r = f(x_0, x_1, \dots, x_{n-1}),$$

design a Boolean expression  $e$  which can compute it.

- **Solution:** ?

Notes:

- **Agenda:** **combinatorial logic** design, where, crucially,
- the output is a function of the input only,
  - computation is viewed as being continuous,
- via coverage of
1. special-purpose design patterns,
  2. special-purpose building blocks, and
  3. general-purpose derivation.

Notes:

## Part 1: special-purpose design patterns

### ► Pattern #1: decomposition.

- Any  $n$ -input,  $m$ -output Boolean function

$$f : \mathbb{B}^n \rightarrow \mathbb{B}^m$$

can be rewritten as  $m$  separate  $n$ -input, 1-output Boolean functions, say

$$\begin{array}{ll} f_0 & : \mathbb{B}^n \rightarrow \mathbb{B} \\ f_1 & : \mathbb{B}^n \rightarrow \mathbb{B} \\ & \vdots \\ f_{m-1} & : \mathbb{B}^n \rightarrow \mathbb{B} \end{array}$$

- As such, we have

$$f(x) \equiv f_0(x) \parallel f_1(x) \parallel \dots \parallel f_{m-1}(x).$$

Notes:

## Part 1: special-purpose design patterns

### ► Pattern #2: sharing.

- Imagine, for example, that we are given a 2-input, 1-bit AND gate.
- If, within some larger circuit, we compute

$$r = x \wedge y$$

and then, somewhere else,

$$r' = x \wedge y$$

then we can replace the two AND gates with one: clearly

$$r = r',$$

so we can share one definition between two usage points.

Notes:

## Part 1: special-purpose design patterns

### ► Pattern #3: independent replication.

- Imagine, for example, that we are given a 2-input, 1-bit AND gate.
- A 2-input,  $m$ -bit AND gate is simply replication of 2-input, 1-bit AND gates, i.e.,

$$r = x \wedge y$$

is computed via

$$r_i = x_i \wedge y_i$$

- for  $0 \leq i < m$ ,
- for  $n = 4$ , as an example, this means

$$\begin{aligned} r_0 &= x_0 \wedge y_0 \\ r_1 &= x_1 \wedge y_1 \\ r_2 &= x_2 \wedge y_2 \\ r_3 &= x_3 \wedge y_3 \end{aligned}$$

Notes:

## Part 1: special-purpose design patterns

### ► Pattern #4: dependent replication.

- Imagine, for example, that we are given a 2-input, 1-bit AND gate.
- An  $n$ -input, 1-bit AND gate is simply replication of 2-input, 1-bit AND gates, i.e.,

$$r = \bigwedge_{i=0}^{n-1} x_i$$

is computed via

$$r = x_0 \wedge (x_1 \wedge \cdots (x_{n-1})),$$

- for  $n = 4$ , as an example, this means

$$\begin{aligned} r &= x_0 \wedge (x_1 \wedge x_2 \wedge x_3) \\ &= x_0 \wedge x_1 \wedge x_2 \wedge x_3 \\ &= (x_0 \wedge x_1) \wedge (x_2 \wedge x_3) \end{aligned}$$

Notes:

## Part 2: special-purpose building blocks (1)

Choice

- **Concept:** the following building blocks can support most forms of choice
    1. a **multiplexer**
      - has  $m$  inputs,
      - has 1 output,
      - uses a  $(\lceil \log_2(m) \rceil)$ -bit control signal input to choose which input is connected to the output,
    - while
    2. a **demultiplexer**
      - has 1 input,
      - has  $m$  outputs,
      - uses a  $(\lceil \log_2(m) \rceil)$ -bit control signal input to choose which output is connected to the input,
- noting that
- the input(s) and output(s) are  $n$ -bit, but clearly must match up,
  - the connection made is continuous, since both components are combinatorial.

Notes:

## Part 2: special-purpose building blocks (2)

Choice

- **Concept:** by analogy,
  1. the C `switch` statement

### Listing

```
1 switch( c ) {  
2   case 0 : r = w; break;  
3   case 1 : r = x; break;  
4   case 2 : r = y; break;  
5   case 3 : r = z; break;  
6 }
```

- acts similarly to a 4-input multiplexer,
2. the C `switch` statement

### Listing

```
1 switch( c ) {  
2   case 0 : r0 = x; break;  
3   case 1 : r1 = x; break;  
4   case 2 : r2 = x; break;  
5   case 3 : r3 = x; break;  
6 }
```

acts similarly to a 4-output demultiplexer.

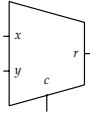
Notes:

## Part 2: special-purpose building blocks (3)

Choice

### Definition

The behaviour of a **2-input, 1-bit multiplexer** component



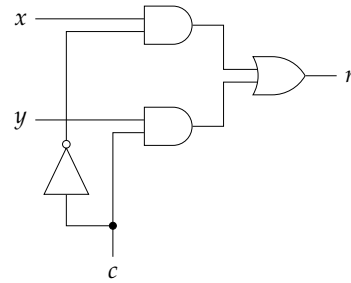
is described by the truth table

$c$	$x$	$y$	$r$
0	0	?	0
0	1	?	1
1	?	0	0
1	?	1	1

which can be used to derive the following implementation:

$$r = (\neg c \wedge x) \vee (c \wedge y)$$

### Circuit



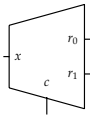
Notes:

## Part 2: special-purpose building blocks (4)

Choice

### Definition

The behaviour of a **2-output, 1-bit demultiplexer** component



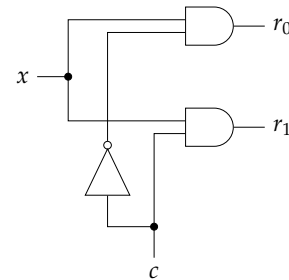
is described by the truth table

$c$	$x$	$r_1$	$r_0$
0	0	?	0
0	1	?	1
1	0	0	?
1	1	1	?

which can be used to derive the following implementation:

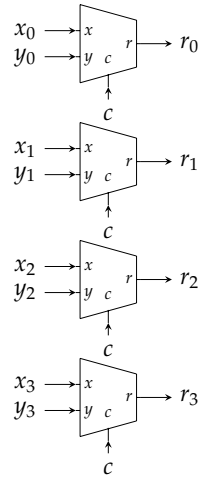
$$\begin{aligned} r_0 &= \neg c \wedge x \\ r_1 &= c \wedge x \end{aligned}$$

### Circuit



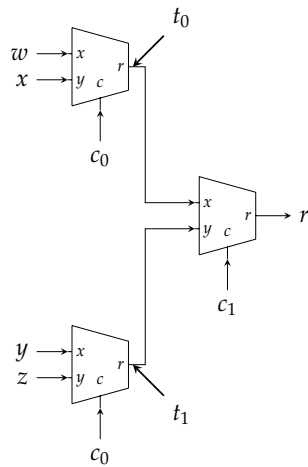
Notes:

### Circuit (2-input, 4-bit multiplexer via independent replication)



Notes:

### Circuit (4-input, 1-bit multiplexer via dependent replication)



Notes:

## Part 2: special-purpose building blocks (7)

### Addition

► **Concept:** the following building blocks can support most forms of arithmetic

1. a **half-adder**

- has 2 inputs:  $x$  and  $y$ ,
- computes the 2-bit result  $x + y$ ,
- has 2 outputs: a sum  $s$ , and a carry-out  $co$  (which are the LSB and MSB of result),

while

2. a **full-adder**

- has 3 inputs:  $x$  and  $y$  plus a carry-in  $ci$ ,
- computes the 2-bit result  $x + y + ci$ ,
- has 2 outputs: a sum  $s$ , and a carry-out  $co$  (which are the LSB and MSB of result),

where all inputs and outputs are 1-bit.

Notes:

## Part 2: special-purpose building blocks (8)

### Addition

#### Definition

The behaviour of a **half-adder** component



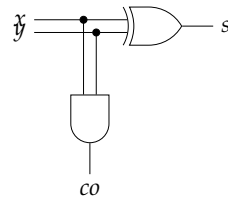
is described by the truth table

$x$	$y$	$co$	$s$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

which can be used to derive the following implementation:

$$\begin{aligned} co &= x \wedge y \\ s &= x \oplus y \end{aligned}$$

#### Circuit



Notes:

Part 2: special-purpose building blocks (9)  
Addition

Definition

The behaviour of a **full-adder** component

ci

co

x

y

is described by the truth table

ci	x	y	co	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

which can be used to derive the following implementation:

co

=

$(x \wedge y) \vee (x \wedge ci) \vee (y \wedge ci)$

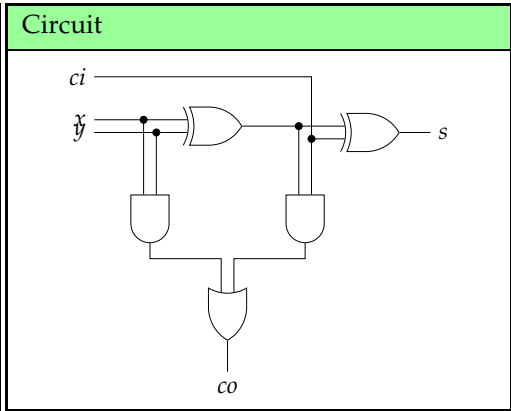
=

$(x \wedge y) \vee ((x \oplus y) \wedge ci)$

s

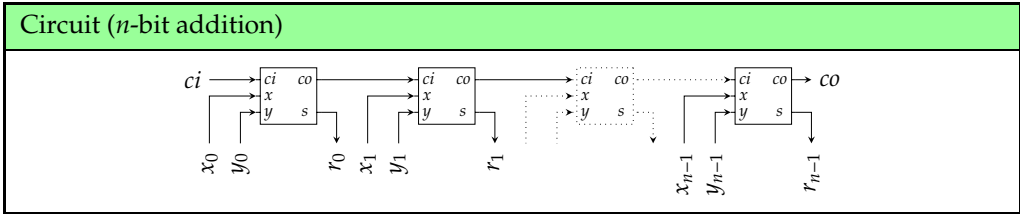
=

$x \oplus y \oplus ci$



Notes:

Part 2: special-purpose building blocks (10)  
Addition



Notes:

## Part 2: special-purpose building blocks (11)

### Comparison

► **Concept:** the following building blocks can support most forms of comparison

1. an **equality comparator**

- has 2 inputs  $x$  and  $y$ ,
- computes the 1 output as

$$r = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

while

2. a **less-than comparator**

- has 2 inputs  $x$  and  $y$ ,
- computes the 1 output as

$$r = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

where all inputs and outputs are 1-bit.

Notes:

## Part 2: special-purpose building blocks (12)

### Comparison

#### Definition

The behaviour of an **equality comparator** component



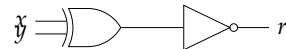
is described by the truth table

$x$	$y$	$r$
0	0	1
0	1	0
1	0	0
1	1	1

which can be used to derive the following implementation:

$$r = \neg(x \oplus y)$$

#### Circuit



Notes:

## Part 2: special-purpose building blocks (13)

### Comparison

#### Definition

The behaviour of a **less-than comparator** component



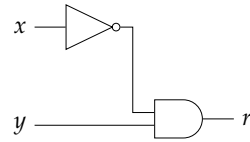
is described by the truth table

$x$	$y$	$r$
0	0	0
0	1	1
1	0	0
1	1	0

which can be used to derive the following implementation:

$$r = \neg x \wedge y$$

#### Circuit

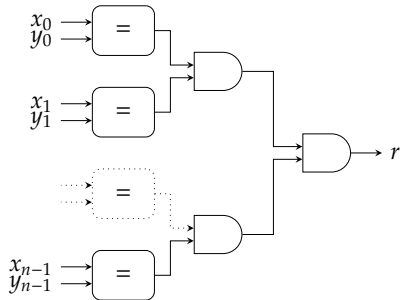


Notes:

## Part 2: special-purpose building blocks (14)

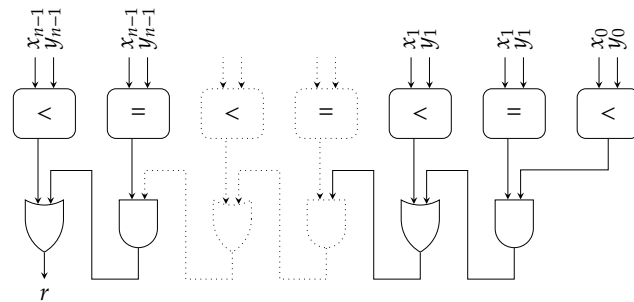
### Comparison

#### Circuit ( $n$ -bit equality comparison)



Notes:

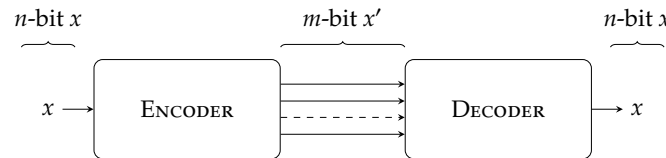
### Circuit ( $n$ -bit less-than comparison)



Notes:

### Part 2: special-purpose building blocks (16) Control

► **Concept:** informally, **encoders** and **decoders** can be viewed as *translators*, i.e.,



or, more formally,

1. an  $n$ -to- $m$  encoder translates an  $n$ -bit input into some  $m$ -bit code word, and
2. an  $m$ -to- $n$  decoder translates an  $m$ -bit code word back into the same  $n$ -bit output

where if only one output (resp. input) is allowed to be 1 at a time, we call it a **one-of-many** encoder (resp. decoder).

Notes:

- A *general* building block is impossible since it depends on the scheme for encoding/decoding: consider an **example** such that
1. to encode, take  $n$  inputs, say  $x_i$  for  $0 \leq i < n$ , and produce a unsigned integer  $x'$  that determines which  $x_i = 1$ ,
  2. to decode, take  $x'$  and set the correct  $x_i = 1$  where for all  $j \neq i$ ,  $x'_j = 0$ .

Notes:

Definition (example encoder)

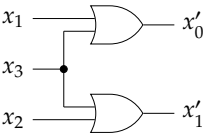
The example encoder is described by the truth table

$x_3$	$x_2$	$x_1$	$x_0$	$x'_1$	$x'_0$
0	0	0	1	0	0
0	0	1	0	0	1
0	1	0	0	1	0
1	0	0	0	1	1

which can be used to derive the following implementation:

$$\begin{aligned} x'_0 &= x_1 \vee x_3 \\ x'_1 &= x_2 \vee x_3 \end{aligned}$$

Circuit (example encoder)



Notes:

### Definition (example decoder)

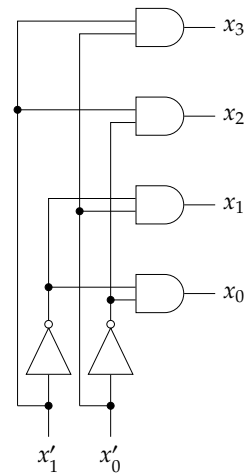
The example decoder is described by the truth table

$x'_1$	$x'_0$	$x_3$	$x_2$	$x_1$	$x_0$
0	0	0	0	0	1
0	1	0	0	1	0
1	0	0	1	0	0
1	1	1	0	0	0

which can be used to derive the following implementation:

$$\begin{aligned}
 x_0 &= \neg x'_0 \wedge \neg x'_1 \\
 x_1 &= x'_0 \wedge \neg x'_1 \\
 x_2 &= \neg x'_0 \wedge x'_1 \\
 x_3 &= x'_0 \wedge x'_1
 \end{aligned}$$

### Circuit (example decoder)



Notes:

- **Problem:** if we break the rules and set both  $x_1 = 1$  and  $x_2 = 1$ , the encoder fails by producing

$$\begin{aligned}
 x'_0 &= x_1 \vee x_3 = 1 \\
 x'_1 &= x_2 \vee x_3 = 1
 \end{aligned}$$

as the result.

- **Solution:** consider a **priority** encoder, where one input is given priority (or preference) over another.

Notes:

Example

Imagine we want to give  $x_j$  priority over each  $x_k$  for  $j > k$ , so  $x_2$  over  $x_1$  and  $x_0$  for example:

$x_3$	$x_2$	$x_1$	$x_0$	$x'_1$	$x'_0$
0	0	0	1	0	0
0	0	1	?	0	1
0	1	?	?	1	0
1	?	?	?	1	1

Now, although potentially  $x_0 = 1$  or  $x_1 = 1$  the output gives priority to  $x_2$ : as long as  $x_2 = 1$  and  $x_3 = 0$ , the output will be  $x'_0 = 0$  and  $x'_1 = 1$  irrespective of  $x_0$  and  $x_1$ .

Notes:

Part 3: general-purpose derivation (1)  
Method #1

Algorithm

**Input:** A truth table for some Boolean function  $f$ , with  $n$  inputs and 1 output

**Output:** A Boolean expression  $e$  that implements  $f$

First let  $I_j$  denote the  $j$ -th input for  $0 \leq j < n$  and  $O$  denote the single output:

1. Find a set  $T$  such that  $i \in T$  iff.  $O = 1$  in the  $i$ -th row of the truth table.
2. For each  $i \in T$ , form a term  $t_i$  by AND'ing together all the variables while following two rules:

2.1 if  $I_j = 1$  in the  $i$ -th row, then we use

$$I_j$$

as is, but

2.2 if  $I_j = 0$  in the  $i$ -th row, then we use

$$\neg I_j.$$

3. An expression implementing the function is then formed by OR'ing together all the terms, i.e.,

$$e = \bigvee_{i \in T} t_i,$$

which is in SoP form.

Notes:

### Example

Consider the example of deriving an expression for XOR, i.e.,

$$r = f(x, y) = x \oplus y,$$

a function described by the following truth table:

$f$		
$x$	$y$	$r$
0	0	0
0	1	1
1	0	1
1	1	0

Notes:

### Example

Consider the example of deriving an expression for XOR, i.e.,

$$r = f(x, y) = x \oplus y,$$

a function described by the following truth table:

$f$		
$x$	$y$	$r$
0	0	0
0	1	1
1	0	1
1	1	0

$\rightsquigarrow i = 1$   
 $\rightsquigarrow i = 2$

Following the algorithm produces:

1. Looking at the truth table, it is clear there are

- ▶  $n = 2$  inputs that we denote  $I_0 = x$  and  $I_1 = y$ , and
- ▶ one output that we denote  $O = r$ .

Clearly  $T = \{1, 2\}$  since  $O = 1$  in rows 1 and 2, while  $O = 0$  in rows 0 and 3.

Notes:

## Part 3: general-purpose derivation (2)

Method #1

### Example

Consider the example of deriving an expression for XOR, i.e.,

$$r = f(x, y) = x \oplus y,$$

a function described by the following truth table:

$f$			
$x$	$y$	$r$	
0	0	0	
0	1	1	$\rightsquigarrow t_1 = \neg x \wedge y$
1	0	1	$\rightsquigarrow t_2 = x \wedge \neg y$
1	1	0	

Following the algorithm produces:

2. Each term  $t_i$  for  $i \in T = \{1, 2\}$  is formed as follows:

- ▶ For  $i = 1$ , we find
  - ▶  $I_0 = x = 0$  and so we use  $\neg x$ ,
  - ▶  $I_1 = y = 1$  and so we use  $y$and hence form the term  $t_1 = \neg x \wedge y$ .
- ▶ For  $i = 2$ , we find
  - ▶  $I_0 = x = 1$  and so we use  $x$ ,
  - ▶  $I_1 = y = 0$  and so we use  $\neg y$and hence form the term  $t_2 = x \wedge \neg y$ .

Notes:

## Part 3: general-purpose derivation (2)

Method #1

### Example

Consider the example of deriving an expression for XOR, i.e.,

$$r = f(x, y) = x \oplus y,$$

a function described by the following truth table:

$f$			
$x$	$y$	$r$	
0	0	0	
0	1	1	$\rightsquigarrow t_1 = \neg x \wedge y$
1	0	1	$\rightsquigarrow t_2 = x \wedge \neg y$
1	1	0	

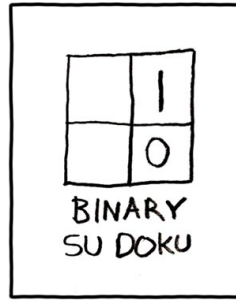
Following the algorithm produces:

3. The expression implementing the function is therefore

$$\begin{aligned} e &= \bigvee_{i \in T} t_i \\ &= \bigvee_{i \in \{1, 2\}} t_i \\ &= (\neg x \wedge y) \vee (x \wedge \neg y) \end{aligned}$$

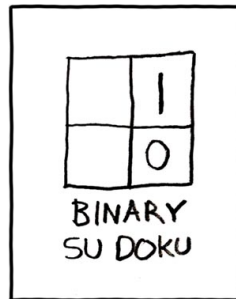
which is in SoP form.

Notes:



Notes:

<https://xkcd.com/74>



Notes:

► Idea:

$$\begin{aligned}(x \wedge y) \vee (x \wedge \neg y) &\equiv x \wedge (y \vee \neg y) && \text{(distribution)} \\ &\equiv x \wedge 1 && \text{(inverse)} \\ &\equiv x && \text{(identity)}\end{aligned}$$

<https://xkcd.com/74>

Algorithm

- Input:** A truth table for some Boolean function  $f$ , with  $n$  inputs and 1 output  
**Output:** A Boolean expression  $e$  that implements  $f$
1. Draw a rectangular ( $p \times q$ )-element grid, st.
    - 1.1  $p \equiv q \equiv 0 \pmod{2}$ , and
    - 1.2  $p \cdot q = 2^n$and each row and column represents one input combination; order rows and columns according to a **Gray code**.
  2. Fill the grid elements with the output corresponding to inputs for that row and column.
  3. Cover rectangular groups of adjacent 1 elements which are of total size  $2^m$  for some  $m$ ; groups can “wrap around” edges of the grid and overlap.
  4. Translate each group into one term of an SoP form Boolean expression  $e$  where
    - 4.1 *bigger* groups, and
    - 4.2 *less* groupsmean a simpler expression.

Notes:

Example

Natural sequence		Gray code sequence	
$\langle 0, 0, 0, 0 \rangle$	$\mapsto 0_{(10)}$	$\langle 0, 0, 0, 0 \rangle$	$\mapsto 0_{(10)}$
$\langle 1, 0, 0, 0 \rangle$	$\mapsto 1_{(10)}$	$\langle 1, 0, 0, 0 \rangle$	$\mapsto 1_{(10)}$
$\langle 0, 1, 0, 0 \rangle$	$\mapsto 2_{(10)}$	$\langle 1, 1, 0, 0 \rangle$	$\mapsto 3_{(10)}$
$\langle 1, 1, 0, 0 \rangle$	$\mapsto 3_{(10)}$	$\langle 0, 1, 0, 0 \rangle$	$\mapsto 2_{(10)}$
$\langle 0, 0, 1, 0 \rangle$	$\mapsto 4_{(10)}$	$\langle 0, 1, 1, 0 \rangle$	$\mapsto 6_{(10)}$
$\langle 1, 0, 1, 0 \rangle$	$\mapsto 5_{(10)}$	$\langle 0, 0, 1, 0 \rangle$	$\mapsto 4_{(10)}$
$\langle 0, 1, 1, 0 \rangle$	$\mapsto 6_{(10)}$	$\langle 1, 0, 1, 0 \rangle$	$\mapsto 5_{(10)}$
$\langle 1, 1, 1, 0 \rangle$	$\mapsto 7_{(10)}$	$\langle 1, 1, 1, 0 \rangle$	$\mapsto 7_{(10)}$
$\vdots$		$\vdots$	

Notes:

Example

Consider an example 4-input, 1-output function:

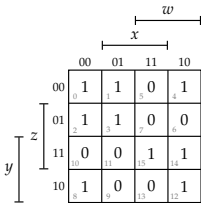
w	x	y	z	r
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

- Notes:
- The first two steps are simple: drawing a grid of an appropriate size then filling it with entries from the truth table are both trivial, even though they still demand some care.
  - Forming the groups is a little harder, in the sense that their “quality” (i.e., number and size) dictates how optimised the resulting expression will be. This example has some non-intuitive cases: the blue group might not be obvious for example, *but* is within the rules (although it is sort of inside-out, it is rectangular and a power-of-two in size).
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Example

Consider an example 4-input, 1-output function:

w	x	y	z	r
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

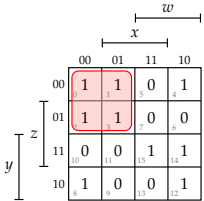


- Notes:
- The first two steps are simple: drawing a grid of an appropriate size then filling it with entries from the truth table are both trivial, even though they still demand some care.
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Consider an example 4-input, 1-output function:

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0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
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0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1



Each group translates into one term of the SoP form expression

$$r = ( \neg w \wedge \neg y )$$

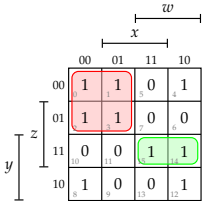
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Consider an example 4-input, 1-output function:

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0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1



Each group translates into one term of the SoP form expression

$$r = ( \neg w \wedge \neg y ) \vee ( w \wedge y \wedge z )$$

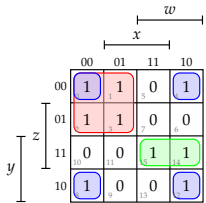
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1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1



Each group translates into one term of the SoP form expression

$$r = ( \neg w \wedge \neg y ) \vee ( w \wedge y \wedge z ) \vee ( \neg x \wedge \neg z )$$

Notes:

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Example

Consider an example 3-input, 1-output function:

x	y	z	r
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	?
1	1	0	1
1	1	1	?

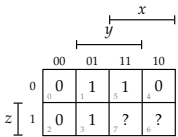
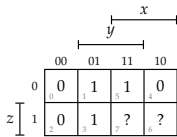
Notes:

- Note the differing shape of this grid versus the previous example: since there are  $n = 3$  variables, we set  $p = 2$  and  $q = 4$  such that the grid contains  $2 \cdot 4 = 2^3 = 8$  cells.
- Adopting the same approach as the previous example, by not ignoring the don't care entries (i.e., assuming they are 0) we have two groups. However, opting to treat one of them as a 1 (which is fine: by definition we don't care what the output is) we only have one:
  - The red group spans column 1 and rows 0 and 1; provided  $x = 0$  and  $y = 1$  we specify *just* those cells, so the expression is  $\neg x \wedge y$ . That is,  $x = 0$  and  $y = 1$  restricts us to column 1 (columns 0, 2 and 3 have at least one of  $x = 1$  or  $y = 0$ ) which is all we need because the group spans *all* rows. Note that the value of  $z$  doesn't matter: cells in the group hold the value 1 regardless of  $z$ .
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<i>x</i>	<i>y</i>	<i>z</i>	<i>r</i>
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	?
1	1	0	1
1	1	1	?



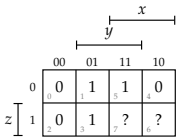
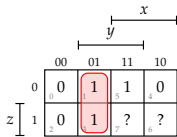
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0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	?
1	1	0	1
1	1	1	?



Each group translates into one term of the SoP form expressions

$$r = (\neg x \wedge y)$$

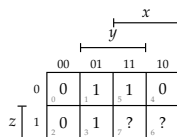
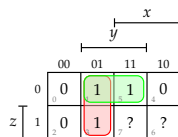
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## Example

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$x$	$y$	$z$	$r$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	?
1	1	0	1
1	1	1	?



Each group translates into one term of the SoP form expressions

$$r = (\neg x \wedge y) \vee (y \wedge \neg z)$$

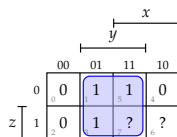
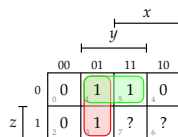
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0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	?
1	1	0	1
1	1	1	?



Each group translates into one term of the SoP form expressions

$$r = (\neg x \wedge y) \vee (y \wedge \neg z)$$

$$r = y$$

where effective use of don't care states yields a clear improvement!

Notes:

- Note the differing shape of this grid versus the previous example: since there are  $n = 3$  variables, we set  $p = 2$  and  $q = 4$  such that the grid contains  $2 \cdot 4 = 2^3 = 8$  cells.
- Adopting the same approach as the previous example, by not ignoring the don't care entries (i.e., assuming they are 0) we have two groups. However, opting to treat one of them as a 1 (which is fine: by definition we don't care what the output is) we only have one:
  - The red group spans column 1 and rows 0 and 1; provided  $x = 0$  and  $y = 1$  we specify *just* those cells, so the expression is  $\neg x \wedge y$ . That is,  $x = 0$  and  $y = 1$  restricts us to column 1 (columns 0, 2 and 3 have at least one of  $x = 1$  or  $y = 0$ ) which is all we need because the group spans *all* rows. Note that the value of  $z$  doesn't matter: cells in the group hold the value 1 regardless of  $z$ .
  - The green group spans columns 1 and 2, in row 0; provided  $y = 1$  and  $z = 0$  we specify *just* those cells, so the expression is  $y \wedge \neg z$ . That is,  $y = 1$  restricts us to columns 1 and 2 (columns 0 and 3 have  $y = 0$ ) and  $z = 0$  restricts us to row 0 (row 1 has  $z = 1$ ). Note that the value of  $x$  doesn't matter: cells in the group hold the value 1 regardless of  $x$ .
  - The blue group spans columns 1 and 2 and rows 0 and 1; provided  $y = 1$  we specify *just* those cells, so the expression is  $y$ . That is,  $y = 1$  restricts us to columns 1 and 2 (columns 0 and 3 have  $y = 0$ ) which is all we need because the group spans *all* rows. Note that the values of  $x$  and  $z$  don't matter: cells in the group hold the value 1 regardless of  $x$  and  $z$ .

Example

Consider an example 3-input, 1-output function:

<i>x</i>	<i>y</i>	<i>z</i>	<i>r</i>
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

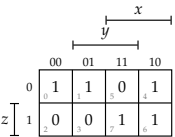
Notes:

- The red group spans columns 2 and 3, in row 1; provided  $x = 1$  and  $z = 1$  we specify *just* those cells, so the expression is  $x \wedge z$ .  $x = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $x = 0$ ) and  $z = 1$  restricts us to row 1 (row 0 has  $z = 0$ ). Note that the value of  $y$  doesn't matter: cells in the group hold the value 1 regardless of  $y$ .
- The green group spans column 3 and rows 0 and 1; provided  $x = 1$  and  $y = 0$  we specify *just* those cells, so the expression is  $x \wedge \neg y$ . That is,  $x = 1$  and  $y = 0$  restricts us to column 3 (columns 0, 1 and 2 have at least one of  $x = 0$  or  $y = 1$ ) which is all we need because the group spans *all* rows. Note that the value of  $z$  doesn't matter: cells in the group hold the value 1 regardless of  $z$ .
- The blue group spans columns 0 and 1, in row 0; provided  $x = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $\neg x \wedge \neg z$ .  $x = 0$  restricts us to columns 0 and 1 (columns 2 and 3 have  $x = 1$ ) and  $z = 0$  restricts us to row 0 (row 1 has  $z = 1$ ). Note that the value of  $y$  doesn't matter: cells in the group hold the value 1 regardless of  $y$ .

Example

Consider an example 3-input, 1-output function:

<i>x</i>	<i>y</i>	<i>z</i>	<i>r</i>
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1



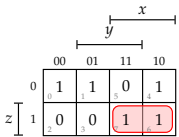
Notes:

- The red group spans columns 2 and 3, in row 1; provided  $x = 1$  and  $z = 1$  we specify *just* those cells, so the expression is  $x \wedge z$ .  $x = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $x = 0$ ) and  $z = 1$  restricts us to row 1 (row 0 has  $z = 0$ ). Note that the value of  $y$  doesn't matter: cells in the group hold the value 1 regardless of  $y$ .
- The green group spans column 3 and rows 0 and 1; provided  $x = 1$  and  $y = 0$  we specify *just* those cells, so the expression is  $x \wedge \neg y$ . That is,  $x = 1$  and  $y = 0$  restricts us to column 3 (columns 0, 1 and 2 have at least one of  $x = 0$  or  $y = 1$ ) which is all we need because the group spans *all* rows. Note that the value of  $z$  doesn't matter: cells in the group hold the value 1 regardless of  $z$ .
- The blue group spans columns 0 and 1, in row 0; provided  $x = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $\neg x \wedge \neg z$ .  $x = 0$  restricts us to columns 0 and 1 (columns 2 and 3 have  $x = 1$ ) and  $z = 0$  restricts us to row 0 (row 1 has  $z = 1$ ). Note that the value of  $y$  doesn't matter: cells in the group hold the value 1 regardless of  $y$ .

Example

Consider an example 3-input, 1-output function:

x	y	z	r
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1



Each group translates into one term of the SoP form expression

$$r = ( \quad x \quad \quad \wedge \quad z \quad )$$

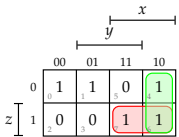
Notes:

- The red group spans columns 2 and 3, in row 1; provided  $x = 1$  and  $z = 1$  we specify *just* those cells, so the expression is  $x \wedge z$ .  $x = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $x = 0$ ) and  $z = 1$  restricts us to row 1 (row 0 has  $z = 0$ ). Note that the value of  $y$  doesn't matter: cells in the group hold the value 1 regardless of  $y$ .
- The green group spans column 3 and rows 0 and 1; provided  $x = 1$  and  $y = 0$  we specify *just* those cells, so the expression is  $x \wedge \neg y$ . That is,  $x = 1$  and  $y = 0$  restricts us to column 3 (columns 0, 1 and 2 have at least one of  $x = 0$  or  $y = 1$ ) which is all we need because the group spans *all* rows. Note that the value of  $z$  doesn't matter: cells in the group hold the value 1 regardless of  $z$ .
- The blue group spans columns 0 and 1, in row 0; provided  $x = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $\neg x \wedge \neg z$ .  $x = 0$  restricts us to columns 0 and 1 (columns 2 and 3 have  $x = 1$ ) and  $z = 0$  restricts us to row 0 (row 1 has  $z = 1$ ). Note that the value of  $y$  doesn't matter: cells in the group hold the value 1 regardless of  $y$ .

Example

Consider an example 3-input, 1-output function:

x	y	z	r
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1



Each group translates into one term of the SoP form expression

$$r = ( \quad x \quad \quad \wedge \quad z \quad ) \vee ( \quad x \quad \wedge \quad \neg y \quad )$$

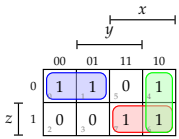
Notes:

- The red group spans columns 2 and 3, in row 1; provided  $x = 1$  and  $z = 1$  we specify *just* those cells, so the expression is  $x \wedge z$ .  $x = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $x = 0$ ) and  $z = 1$  restricts us to row 1 (row 0 has  $z = 0$ ). Note that the value of  $y$  doesn't matter: cells in the group hold the value 1 regardless of  $y$ .
- The green group spans column 3 and rows 0 and 1; provided  $x = 1$  and  $y = 0$  we specify *just* those cells, so the expression is  $x \wedge \neg y$ . That is,  $x = 1$  and  $y = 0$  restricts us to column 3 (columns 0, 1 and 2 have at least one of  $x = 0$  or  $y = 1$ ) which is all we need because the group spans *all* rows. Note that the value of  $z$  doesn't matter: cells in the group hold the value 1 regardless of  $z$ .
- The blue group spans columns 0 and 1, in row 0; provided  $x = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $\neg x \wedge \neg z$ .  $x = 0$  restricts us to columns 0 and 1 (columns 2 and 3 have  $x = 1$ ) and  $z = 0$  restricts us to row 0 (row 1 has  $z = 1$ ). Note that the value of  $y$  doesn't matter: cells in the group hold the value 1 regardless of  $y$ .

Example

Consider an example 3-input, 1-output function:

<i>x</i>	<i>y</i>	<i>z</i>	<i>r</i>
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1



Each group translates into one term of the SoP form expression

$$r = ( \begin{matrix} x \\ x \\ \neg x \end{matrix} \wedge \begin{matrix} z \\ \neg y \\ \neg z \end{matrix} ) \vee$$

Notes:

- The red group spans columns 2 and 3, in row 1; provided  $x = 1$  and  $z = 1$  we specify *just* those cells, so the expression is  $x \wedge z$ .  $x = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $x = 0$ ) and  $z = 1$  restricts us to row 1 (row 0 has  $z = 0$ ). Note that the value of  $y$  doesn't matter: cells in the group hold the value 1 regardless of  $y$ .
- The green group spans column 3 and rows 0 and 1; provided  $x = 1$  and  $y = 0$  we specify *just* those cells, so the expression is  $x \wedge \neg y$ . That is,  $x = 1$  and  $y = 0$  restricts us to column 3 (columns 0, 1 and 2 have at least one of  $x = 0$  or  $y = 1$ ) which is all we need because the group spans *all* rows. Note that the value of  $z$  doesn't matter: cells in the group hold the value 1 regardless of  $z$ .
- The blue group spans columns 0 and 1, in row 0; provided  $x = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $\neg x \wedge \neg z$ .  $x = 0$  restricts us to columns 0 and 1 (columns 2 and 3 have  $x = 1$ ) and  $z = 0$  restricts us to row 0 (row 1 has  $z = 1$ ). Note that the value of  $y$  doesn't matter: cells in the group hold the value 1 regardless of  $y$ .

Example

Consider an example 4-input, 2-output function:

<i>w</i>	<i>x</i>	<i>y</i>	<i>z</i>	<i>r</i> <sub>1</sub>	<i>r</i> <sub>0</sub>
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	?	?
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	?	?

Notes:

- The red group spans columns 2 and 3 in row 0; provided  $w = 1$ ,  $y = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $w \wedge \neg y \wedge \neg z$ . That is,  $w = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $w = 0$ ) and  $y = 0$  and  $z = 0$  restricts us to row 0 (rows 1, 2 and 3 have at least one of  $y = 1$  or  $z = 1$ ). Note that the value of  $x$  doesn't matter: cells in the group hold the value 1 regardless of  $x$ .
- The green group spans column 1 and rows 2 and 3; provided  $w = 0$ ,  $x = 0$  and  $y = 1$  we specify *just* those cells, so the expression is  $\neg w \wedge \neg x \wedge y$ . That is,  $w = 0$  and  $x = 0$  restricts us to column 0 (columns 1, 2 and 3 have at least one of  $w = 1$  or  $x = 1$ ) and  $y = 1$  restricts us to rows 2 and 3 (rows 0 and 1 have  $y = 0$ ). Note that the value of  $z$  doesn't matter: cells in the group hold the value 1 regardless of  $z$ .
- The blue group spans columns 1 and 2 and rows 1 and 2; provided  $x = 1$  and  $z = 1$  we specify *just* those cells, so the expression is  $x \wedge z$ . That is,  $x = 1$  restricts us to columns 1 and 2 (columns 0 and 3 have  $x = 0$ ) and  $z = 1$  restricts us to rows 1 and 2 (rows 0 and 3 have  $z = 0$ ). Note that the value of  $w$  and  $y$  don't matter: cells in the group hold the value 1 regardless of  $w$  and  $y$ .
- The magenta group spans columns 1 and 2 in row 0; provided  $x = 1$ ,  $y = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $x \wedge \neg y \wedge \neg z$ . That is,  $x = 1$  restricts us to columns 1 and 2 (columns 0 and 3 have  $x = 0$ ) and  $y = 0$  and  $z = 0$  restricts us to row 0 (rows 1, 2 and 3 have at least one of  $y = 1$  or  $z = 1$ ). Note that the value of  $w$  doesn't matter: cells in the group hold the value 1 regardless of  $w$ .
- The yellow group spans column 0 and rows 1 and 2; provided  $w = 0$ ,  $x = 0$  and  $z = 1$  we specify *just* those cells, so the expression is  $\neg w \wedge \neg x \wedge z$ . That is,  $w = 0$  and  $x = 0$  restricts us to column 0 (columns 1, 2 and 3 have at least one of  $w = 1$  or  $x = 1$ ) and  $y = 1$  restricts us to rows 1 and 2 (rows 0 and 1 have  $y = 0$ ). Note that the value of  $w$  doesn't matter: cells in the group hold the value 1 regardless of  $w$ .
- The orange group spans columns 2 and 3 and rows 2 and 3; provided  $w = 1$  and  $y = 1$  we specify *just* those cells, so the expression is  $w \wedge y$ . That is,  $w = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $w = 0$ ) and  $y = 1$  restricts us to rows 2 and 3 (rows 0 and 1 have  $y = 0$ ). Note that the value of  $x$  and  $z$  don't matter: cells in the group hold the value 1 regardless of  $x$  and  $z$ .

## Example

Consider an example 4-input, 2-output function:

$w$	$x$	$y$	$z$	$r_1$	$r_0$
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	?	?
0	1	0	0	?	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	?	?

		$x$				$w$	
		00	01	11	10		
$r_1$	00	0	0	?	1		
	01	0	1	?	0		
	11	?	?	?	?		
	10	1	0	?	0		
						$z$	$y$

		$x$				$w$	
		00	01	11	10		
$r_0$	00	0	1	?	0		
	01	1	0	?	0		
	11	?	?	?	?		
	10	0	0	?	1		
						$z$	$y$

Notes:

- The red group spans columns 2 and 3 in row 0; provided  $w = 1, y = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $w \wedge \neg y \wedge \neg z$ . That is,  $w = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $w = 0$ ) and  $y = 0$  and  $z = 0$  restricts us to row 0 (rows 1, 2 and 3 have at least one of  $y = 1$  or  $z = 1$ ). Note that the value of  $x$  doesn't matter: cells in the group hold the value 1 regardless of  $x$ .
  - The green group spans column 1 and rows 2 and 3; provided  $w = 0, x = 0$  and  $y = 1$  we specify *just* those cells, so the expression is  $\neg w \wedge \neg x \wedge y$ . That is,  $w = 0$  and  $x = 0$  restricts us to column 0 (columns 1, 2 and 3 have at least one of  $w = 1$  or  $x = 1$ ) and  $y = 1$  restricts us to rows 2 and 3 (rows 0 and 1 have  $y = 0$ ). Note that the value of  $z$  doesn't matter: cells in the group hold the value 1 regardless of  $z$ .
  - The blue group spans columns 1 and 2 and rows 1 and 2; provided  $x = 1$  and  $z = 1$  we specify *just* those cells, so the expression is  $x \wedge z$ . That is,  $x = 1$  restricts us to columns 1 and 2 (columns 0 and 3 have  $x = 0$ ) and  $z = 1$  restricts us to rows 1 and 2 (rows 0 and 3 have  $z = 0$ ). Note that the value of  $w$  and  $y$  don't matter: cells in the group hold the value 1 regardless of  $w$  and  $y$ .
  - The magenta group spans columns 1 and 2 in row 0; provided  $x = 1, y = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $x \wedge \neg y \wedge \neg z$ . That is,  $x = 1$  restricts us to columns 1 and 2 (columns 0 and 3 have  $x = 0$ ) and  $y = 0$  and  $z = 0$  restricts us to row 0 (rows 1, 2 and 3 have at least one of  $y = 1$  or  $z = 1$ ). Note that the value of  $w$  doesn't matter: cells in the group hold the value 1 regardless of  $w$ .
  - The yellow group spans column 0 and rows 1 and 2; provided  $w = 0, x = 0$  and  $z = 1$  we specify *just* those cells, so the expression is  $\neg w \wedge \neg x \wedge z$ . That is,  $w = 0$  and  $x = 0$  restricts us to column 0 (columns 1, 2 and 3 have at least one of  $w = 1$  or  $x = 1$ ) and  $y = 1$  restricts us to rows 1 and 2 (rows 0 and 1 have  $y = 0$ ). Note that the value of  $w$  doesn't matter: cells in the group hold the value 1 regardless of  $w$ .
  - The orange group spans columns 2 and 3 and rows 2 and 3; provided  $w = 1$  and  $y = 1$  we specify *just* those cells, so the expression is  $w \wedge y$ . That is,  $w = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $w = 0$ ) and  $y = 1$  restricts us to rows 2 and 3 (rows 0 and 1 have  $y = 0$ ). Note that the value of  $x$  and  $z$  don't matter: cells in the group hold the value 1 regardless of  $x$  and  $z$ .

## Example

Consider an example 4-input, 2-output function:

$w$	$x$	$y$	$z$	$r_1$	$r_0$
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	?	?
0	1	0	0	?	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	?	?

		$x$				$w$	
		00	01	11	10		
$r_1$	00	0	0	?	1		
	01	0	1	?	0		
	11	?	?	?	?		
	10	1	0	?	0		
						$z$	$y$

		$x$				$w$	
		00	01	11	10		
$r_0$	00	0	1	?	0		
	01	1	0	?	0		
	11	?	?	?	?		
	10	0	0	?	1		
						$z$	$y$

Each group translates into one term of the SoP form expressions

$$r_1 = ( w \wedge \neg y \wedge \neg z )$$

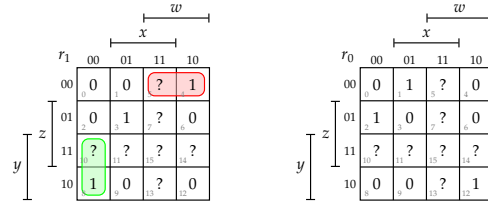
Notes:

- The red group spans columns 2 and 3 in row 0; provided  $w = 1, y = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $w \wedge \neg y \wedge \neg z$ . That is,  $w = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $w = 0$ ) and  $y = 0$  and  $z = 0$  restricts us to row 0 (rows 1, 2 and 3 have at least one of  $y = 1$  or  $z = 1$ ). Note that the value of  $x$  doesn't matter: cells in the group hold the value 1 regardless of  $x$ .
  - The green group spans column 1 and rows 2 and 3; provided  $w = 0, x = 0$  and  $y = 1$  we specify *just* those cells, so the expression is  $\neg w \wedge \neg x \wedge y$ . That is,  $w = 0$  and  $x = 0$  restricts us to column 0 (columns 1, 2 and 3 have at least one of  $w = 1$  or  $x = 1$ ) and  $y = 1$  restricts us to rows 2 and 3 (rows 0 and 1 have  $y = 0$ ). Note that the value of  $z$  doesn't matter: cells in the group hold the value 1 regardless of  $z$ .
  - The blue group spans columns 1 and 2 and rows 1 and 2; provided  $x = 1$  and  $z = 1$  we specify *just* those cells, so the expression is  $x \wedge z$ . That is,  $x = 1$  restricts us to columns 1 and 2 (columns 0 and 3 have  $x = 0$ ) and  $z = 1$  restricts us to rows 1 and 2 (rows 0 and 3 have  $z = 0$ ). Note that the value of  $w$  and  $y$  don't matter: cells in the group hold the value 1 regardless of  $w$  and  $y$ .
  - The magenta group spans columns 1 and 2 in row 0; provided  $x = 1, y = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $x \wedge \neg y \wedge \neg z$ . That is,  $x = 1$  restricts us to columns 1 and 2 (columns 0 and 3 have  $x = 0$ ) and  $y = 0$  and  $z = 0$  restricts us to row 0 (rows 1, 2 and 3 have at least one of  $y = 1$  or  $z = 1$ ). Note that the value of  $w$  doesn't matter: cells in the group hold the value 1 regardless of  $w$ .
  - The yellow group spans column 0 and rows 1 and 2; provided  $w = 0, x = 0$  and  $z = 1$  we specify *just* those cells, so the expression is  $\neg w \wedge \neg x \wedge z$ . That is,  $w = 0$  and  $x = 0$  restricts us to column 0 (columns 1, 2 and 3 have at least one of  $w = 1$  or  $x = 1$ ) and  $y = 1$  restricts us to rows 1 and 2 (rows 0 and 1 have  $y = 0$ ). Note that the value of  $w$  doesn't matter: cells in the group hold the value 1 regardless of  $w$ .
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Consider an example 4-input, 2-output function:

$w$	$x$	$y$	$z$	$r_1$	$r_0$
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0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	?	?
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	?	?



Each group translates into one term of the SoP form expressions

$$r_1 = (w \wedge \neg y \wedge \neg z) \vee (y \wedge \neg w \wedge \neg x)$$

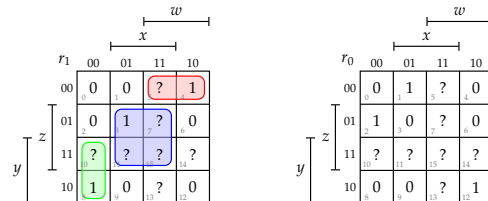
Notes:

- The red group spans columns 2 and 3 in row 0; provided  $w = 1, y = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $w \wedge \neg y \wedge \neg z$ . That is,  $w = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $w = 0$ ) and  $y = 0$  and  $z = 0$  restricts us to row 0 (rows 1, 2 and 3 have at least one of  $y = 1$  or  $z = 1$ ). Note that the value of  $x$  doesn't matter: cells in the group hold the value 1 regardless of  $x$ .
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0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
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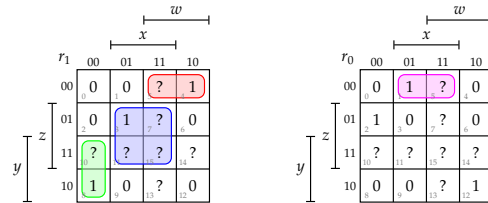
Notes:

- The red group spans columns 2 and 3 in row 0; provided  $w = 1, y = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $w \wedge \neg y \wedge \neg z$ . That is,  $w = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $w = 0$ ) and  $y = 0$  and  $z = 0$  restricts us to row 0 (rows 1, 2 and 3 have at least one of  $y = 1$  or  $z = 1$ ). Note that the value of  $x$  doesn't matter: cells in the group hold the value 1 regardless of  $x$ .
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0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
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$$r_1 = ( \begin{matrix} w & \wedge & \neg y & \wedge & \neg z \\ y & \wedge & \neg w & \wedge & \neg x \\ x & \wedge & z \end{matrix} ) \vee$$

$$r_0 = ( \begin{matrix} x & \wedge & \neg y & \wedge & \neg z \end{matrix} )$$

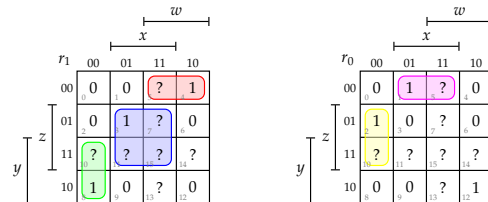
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- The red group spans columns 2 and 3 in row 0; provided  $w = 1, y = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $w \wedge \neg y \wedge \neg z$ . That is,  $w = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $w = 0$ ) and  $y = 0$  and  $z = 0$  restricts us to row 0 (rows 1, 2 and 3 have at least one of  $y = 1$  or  $z = 1$ ). Note that the value of  $x$  doesn't matter: cells in the group hold the value 1 regardless of  $x$ .
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0	0	0	1	0	1
0	0	1	0	1	0
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0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
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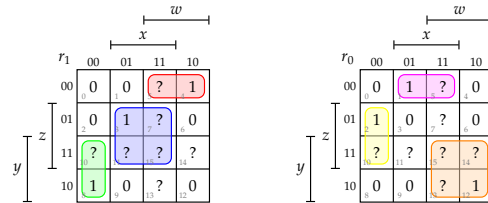
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0	0	1	0	1	0
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0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
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$$r_0 = (x \wedge \neg y \wedge \neg z) \vee (z \wedge \neg w \wedge \neg x) \vee (w \wedge y)$$

Notes:

- The red group spans columns 2 and 3 in row 0; provided  $w = 1, y = 0$  and  $z = 0$  we specify *just* those cells, so the expression is  $w \wedge \neg y \wedge \neg z$ . That is,  $w = 1$  restricts us to columns 2 and 3 (columns 0 and 1 have  $w = 0$ ) and  $y = 0$  and  $z = 0$  restricts us to row 0 (rows 1, 2 and 3 have at least one of  $y = 1$  or  $z = 1$ ). Note that the value of  $x$  doesn't matter: cells in the group hold the value 1 regardless of  $x$ .
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## Conclusions

### ► Take away points:

- There are a *huge* number of challenges, even with (relatively) simple problems, e.g.,
  - how do we describe what the design should do?
  - how do we structure the design?
  - what sort of standard cell library do we use?
  - do we aim for the fewest gates?
  - do we aim for shortest critical path?
  - how do we cope with propagation delay and fan-out?
  - ...
- The three themes we've covered, i.e.,
  - high-level design patterns,
  - low-level, mechanical derivation and optimisation of Boolean expressions,
  - building-block components,
 allows us to address such challenges: in combination, they support development of effective (combinatorial) design and implementation.
- In many cases, use of appropriate **Electronic Design Automation (EDA)** tools can provide (semi-)automatic solutions.

Notes:

## Additional Reading

- ▶ *Wikipedia: Combinational logic*. URL: [https://en.wikipedia.org/wiki/Combinational\\_logic](https://en.wikipedia.org/wiki/Combinational_logic).
- ▶ D. Page. “Chapter 2: Basics of digital logic”. In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009.
- ▶ W. Stallings. “Chapter 11: Digital logic”. In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013.
- ▶ A.S. Tanenbaum and T. Austin. “Section 3.2.2: Combinatorial circuits”. In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012.

Notes:

## References

- [1] *Wikipedia: Combinational logic*. URL: [https://en.wikipedia.org/wiki/Combinational\\_logic](https://en.wikipedia.org/wiki/Combinational_logic) (see p. 125).
- [2] D. Page. “Chapter 2: Basics of digital logic”. In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009 (see p. 125).
- [3] W. Stallings. “Chapter 11: Digital logic”. In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013 (see p. 125).
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Notes: