- Concept: consider

$$\begin{aligned} \hat{x} &\mapsto x \\ \hat{y} &\mapsto y \end{aligned}$$

- Concept: consider

$$\begin{aligned} \hat{x} &\mapsto x \\ \hat{y} &\mapsto y \\ r &= x \times y \end{aligned}$$

- Concept: consider

$$\begin{array}{rcccl} & \hat{x} & \mapsto & x & \\ & \hat{y} & \mapsto & y & \\ f(\hat{x},\hat{y}) = & \hat{r} & \mapsto & r & = x \times y \end{array}$$

where $f$

1. has an action on $\hat{x}$ and $\hat{y}$ compatible with that of $\times$ on $x$ and $y$:
   - accepts $n$-bit
     - **multiplier** $\hat{y}$ (that "does the multiplying"), and
     - **multiplicand** $\hat{x}$ (that "is multiplied")

     as input, and
   - produces an $(2 \cdot n)$-bit **product** $\hat{r}$ as output,
2. is a Boolean function:

$$f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^{2 \cdot n}$$

- Agenda: produce a design(s) for $f$, which
  1. functions correctly, and
  2. satisfies pertinent quality metrics (e.g., is efficient in time and/or space).

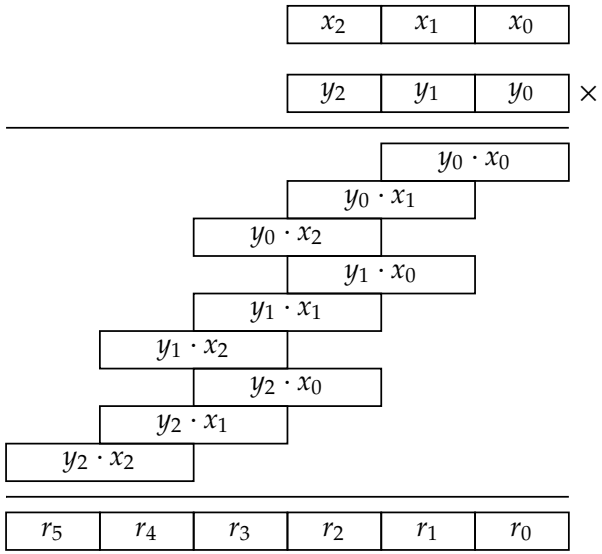
# Part 1: multiplication in theory (1)

## Example

$$
\begin{array}{lrcccccc}
x = & 623_{(10)} & \mapsto & & & 6 & 2 & 3 \\
y = & 567_{(10)} & \mapsto & & & 5 & 6 & 7 \quad \times \\
\hline
p_0 = 7 \cdot 3 \cdot 10^0 = & 21_{(10)} & \mapsto & & & & 2 & 1 \\
p_1 = 7 \cdot 2 \cdot 10^1 = & 140_{(10)} & \mapsto & & & 1 & 4 & \\
p_2 = 7 \cdot 6 \cdot 10^2 = & 4200_{(10)} & \mapsto & & 4 & 2 & & \\
p_3 = 6 \cdot 3 \cdot 10^1 = & 180_{(10)} & \mapsto & & & 1 & 8 & \\
p_4 = 6 \cdot 2 \cdot 10^2 = & 1200_{(10)} & \mapsto & & 1 & 2 & & \\
p_5 = 6 \cdot 6 \cdot 10^3 = & 36000_{(10)} & \mapsto & 3 & 6 & & & \\
p_6 = 5 \cdot 3 \cdot 10^2 = & 1500_{(10)} & \mapsto & & 1 & 5 & & \\
p_7 = 5 \cdot 2 \cdot 10^3 = & 10000_{(10)} & \mapsto & 1 & 0 & & & \\
p_8 = 5 \cdot 6 \cdot 10^4 = & 300000_{(10)} & \mapsto & 3 & 0 & & & \\
\hline
r = & 353241_{(10)} & \mapsto & 3 & 5 & 3 & 2 & 4 \quad 1
\end{array}
$$

## Example (operand scanning)

Consider an example where where $|x| = |y| = 3$:

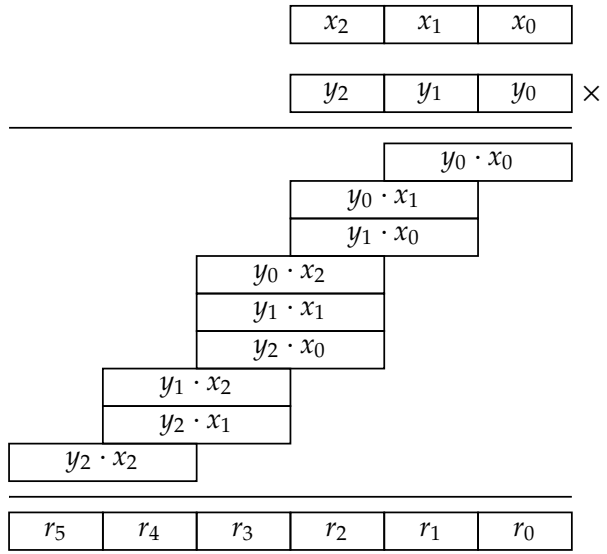



Notice that

1. an outer-loop steps through digits of $y$, say $y_j$,
2. an inner-loop steps through digits of $x$, say $x_i$.

## Example (product scanning)

Consider an example where where $|x| = |y| = 3$:




Notice that

1. an outer-loop steps through digits of $r$, say $r_k$,
2. two inner-loops step through matching digits of $y$ and $x$, say $y_j$ and $x_i$.

## Algorithm (operand scanning)

**Input:** Two unsigned, base-$b$ integers $x$ and $y$
**Output:** An unsigned, base-$b$ integer $r = x \cdot y$

1 $l_x \leftarrow |x|, l_y \leftarrow |y|, l_r \leftarrow l_x + l_y$
2 $r \leftarrow 0$
3 **for** $j = 0$ **upto** $l_y - 1$ **step** $+1$ **do**
4 $\quad c \leftarrow 0$
5 $\quad$ **for** $i = 0$ **upto** $l_x - 1$ **step** $+1$ **do**
6 $\quad\quad u \cdot b + v = t \leftarrow y_j \cdot x_i + r_{j+i} + c$
7 $\quad\quad r_{j+i} \leftarrow v$
8 $\quad\quad c \leftarrow u$
9 $\quad$ **end**
10 $\quad r_{j+l_x} \leftarrow c$
11 **end**
12 **return** $r$

## Example (operand scanning)

Consider a case where $b = 10$, $x = 623_{(10)}$ and $y = 567_{(10)}$:

| $j$ | $i$ | $r$ | $c$ | $y_i$ | $x_j$ | $t = y_i \cdot x_i + r_{i+j} + c$ | $r'$ | $c'$ |
|---|---|---|---|---|---|---|---|---|
| | | $\langle 0,0,0,0,0,0 \rangle$ | | | | | | |
| 0 | 0 | $\langle 0,0,0,0,0,0 \rangle$ | 0 | 7 | 3 | 21 | $\langle 1,0,0,0,0,0 \rangle$ | 2 |
| 0 | 1 | $\langle 1,0,0,0,0,0 \rangle$ | 2 | 7 | 2 | 16 | $\langle 1,6,0,0,0,0 \rangle$ | 1 |
| 0 | 2 | $\langle 1,6,0,0,0,0 \rangle$ | 1 | 7 | 6 | 43 | $\langle 1,6,3,0,0,0 \rangle$ | 4 |
| 0 | | $\langle 1,6,3,0,0,0 \rangle$ | 4 | | | | $\langle 1,6,3,4,0,0 \rangle$ | |
| 1 | 0 | $\langle 1,6,3,4,0,0 \rangle$ | 0 | 6 | 3 | 24 | $\langle 1,4,3,4,0,0 \rangle$ | 2 |
| 1 | 1 | $\langle 1,4,3,4,0,0 \rangle$ | 2 | 6 | 2 | 17 | $\langle 1,4,7,4,0,0 \rangle$ | 1 |
| 1 | 2 | $\langle 1,4,7,4,0,0 \rangle$ | 1 | 6 | 6 | 41 | $\langle 1,4,7,1,0,0 \rangle$ | 4 |
| 1 | | $\langle 1,4,7,1,0,0 \rangle$ | 4 | | | | $\langle 1,4,7,1,4,0 \rangle$ | |
| 2 | 0 | $\langle 1,4,7,1,4,0 \rangle$ | 0 | 5 | 3 | 22 | $\langle 1,4,2,1,4,0 \rangle$ | 2 |
| 2 | 1 | $\langle 1,4,2,1,4,0 \rangle$ | 2 | 5 | 2 | 13 | $\langle 1,4,2,3,4,0 \rangle$ | 1 |
| 2 | 2 | $\langle 1,4,2,3,5,0 \rangle$ | 1 | 5 | 6 | 35 | $\langle 1,4,2,3,5,0 \rangle$ | 3 |
| 2 | | $\langle 1,4,2,3,5,0 \rangle$ | 3 | | | | $\langle 1,4,2,3,5,3 \rangle$ | 3 |
| | | $\langle 1,4,2,3,5,3 \rangle$ | | | | | | |

## Algorithm (product scanning)

**Input:** Two unsigned, base-$b$ integers $x$ and $y$
**Output:** An unsigned, base-$b$ integer $r = x \cdot y$

```
1  l_x ← |x|, l_y ← |y|, l_r ← l_x + l_y
2  r ← 0, c_0 ← 0, c_1 ← 0, c_2 ← 0
3  for k = 0 upto l_x + l_y − 1 step +1 do
4  |  for j = 0 upto l_y − 1 step +1 do
5  |  |  for i = 0 upto l_x − 1 step +1 do
6  |  |  |  if (j + i) = k then
7  |  |  |  |  u · b + v = t ← y_j · x_i
8  |  |  |  |  c · b + c_0 = t ← c_0 + v
9  |  |  |  |  c · b + c_1 = t ← c_1 + u + c
10 |  |  |  |  c_2 ← c_2 + c
11 |  |  |  end
12 |  |  end
13 |  end
14 |  r_k ← c_0, c_0 ← c_1, c_1 ← c_2, c_2 ← 0
15 end
16 r_{l_x+l_y−1} ← c_0
```

## Example (product scanning)

Consider a case where $b = 10$, $x = 623_{(10)}$ and $y = 567_{(10)}$:

| $k$ | $j$ | $i$ | $r$ | $c_2$ | $c_1$ | $c_0$ | $y_i$ | $x_j$ | $t = y_i \cdot x_i$ | $r'$ | $c'_2$ | $c'_1$ | $c'_0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\langle 0,0,0,0,0,0 \rangle$ | 0 | 0 | 0 | | | | | | | |
| 0 | 0 | 0 | $\langle 0,0,0,0,0,0 \rangle$ | 0 | 0 | 0 | 7 | 3 | 21 | $\langle 0,0,0,0,0,0 \rangle$ | 0 | 2 | 1 |
| 0 | | | $\langle 0,0,0,0,0,0 \rangle$ | 0 | 2 | 1 | | | | $\langle 1,0,0,0,0,0 \rangle$ | 0 | 0 | 2 |
| 1 | 0 | 1 | $\langle 1,0,0,0,0,0 \rangle$ | 0 | 0 | 2 | 7 | 2 | 14 | $\langle 1,0,0,0,0,0 \rangle$ | 0 | 1 | 6 |
| 1 | 1 | 0 | $\langle 1,0,0,0,0,0 \rangle$ | 0 | 1 | 6 | 6 | 3 | 18 | $\langle 1,0,0,0,0,0 \rangle$ | 0 | 3 | 4 |
| 1 | | | $\langle 1,0,0,0,0,0 \rangle$ | 0 | 3 | 4 | | | | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 0 | 3 |
| 2 | 0 | 2 | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 0 | 3 | 7 | 6 | 42 | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 4 | 5 |
| 2 | 1 | 1 | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 4 | 5 | 6 | 2 | 12 | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 5 | 7 |
| 2 | 2 | 0 | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 4 | 7 | 5 | 3 | 15 | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 7 | 2 |
| 2 | | | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 7 | 2 | | | | $\langle 1,4,2,0,0,0 \rangle$ | 0 | 0 | 7 |
| 3 | 1 | 2 | $\langle 1,4,2,0,0,0 \rangle$ | 0 | 0 | 7 | 6 | 6 | 36 | $\langle 1,4,2,0,0,0 \rangle$ | 0 | 4 | 3 |
| 3 | 2 | 1 | $\langle 1,4,2,0,0,0 \rangle$ | 0 | 4 | 3 | 5 | 2 | 10 | $\langle 1,4,2,0,0,0 \rangle$ | 0 | 5 | 3 |
| 3 | | | $\langle 1,4,2,0,0,0 \rangle$ | 0 | 5 | 3 | | | | $\langle 1,4,2,3,0,0 \rangle$ | 0 | 0 | 5 |
| 4 | 2 | 2 | $\langle 1,4,2,3,0,0 \rangle$ | 0 | 0 | 5 | 5 | 6 | 30 | $\langle 1,4,2,3,0,0 \rangle$ | 0 | 3 | 5 |
| 4 | | | $\langle 1,4,2,3,0,0 \rangle$ | 0 | 3 | 5 | | | | $\langle 1,4,2,3,5,0 \rangle$ | 0 | 0 | 3 |
| | | | $\langle 1,4,2,3,5,0 \rangle$ | 0 | 0 | 3 | | | | $\langle 1,4,2,3,5,3 \rangle$ | 0 | 0 | 3 |
| | | | $\langle 1,4,2,3,5,3 \rangle$ | | | | | | | | | | |

- Idea:
  - multiplication *means* repeated addition, i.e.,

$$y \times x = \underbrace{x + x + \cdots + x}_{y \text{ terms}},$$

  so if $y = 14_{(10)}$ we have

$$y \times x = x + x + x + x + x + x + x + x + x + x + x + x + x + x.$$

  - expressing $y$ in base-2, we can rewrite this as

$$\begin{aligned} y \times x &= \left(\sum_{i=0}^{n-1} y_i \cdot 2^i\right) \times x \\ &= (y_0 \cdot 2^0 + y_1 \cdot 2^1 + \cdots + y_{n-1} \cdot 2^{n-1}) \times x \\ &= (y_0 \cdot 2^0 \cdot x) + (y_1 \cdot 2^1 \cdot x) + \cdots + (y_{n-1} \cdot 2^{n-1} \cdot x) \\ &= (y_0 \cdot x \cdot 2^0) + (y_1 \cdot x \cdot 2^1) + \cdots + (y_{n-1} \cdot x \cdot 2^{n-1}) \end{aligned}$$

# Part 2: multiplication in practice: an algorithm (5)

## Repeated addition

- Idea:
  - given $y = 14_{(10)} = 1110_{(2)}$ we can see that

$$\begin{aligned} y \times x &= y_0 \cdot x \cdot 2^0 + y_1 \cdot x \cdot 2^1 + y_2 \cdot x \cdot 2^2 + y_3 \cdot x \cdot 2^3 \\ &= 0 \cdot x \cdot 2^0 + 1 \cdot x \cdot 2^1 + 1 \cdot x \cdot 2^2 + 1 \cdot x \cdot 2^3 \\ &= 0 \cdot x + 2 \cdot x + 4 \cdot x + 8 \cdot x \\ &= 14 \cdot x \end{aligned}$$

  - given $y = 14_{(10)} = 1110_{(2)}$ we can see that

$$\begin{aligned} y \times x &= y_0 \cdot x + 2 \cdot ( y_1 \cdot x + 2 \cdot ( y_2 \cdot x + 2 \cdot ( y_3 \cdot x + 2 \cdot ( 0 ) ) ) ) \\ &= 0 \cdot x + 2 \cdot ( 1 \cdot x + 2 \cdot ( 1 \cdot x + 2 \cdot ( 1 \cdot x + 2 \cdot ( 0 ) ) ) ) \\ &= 0 \cdot x + 2 \cdot ( 1 \cdot x + 2 \cdot ( 1 \cdot x + 2 \cdot ( 1 \cdot x + 0 ) ) ) \\ &= 0 \cdot x + 2 \cdot ( 1 \cdot x + 2 \cdot ( 1 \cdot x + 2 \cdot ( 1 \cdot x ) ) ) \\ &= 0 \cdot x + 2 \cdot ( 1 \cdot x + 2 \cdot ( 1 \cdot x + 2 \cdot x ) ) \\ &= 0 \cdot x + 2 \cdot ( 1 \cdot x + 2 \cdot ( 3 \cdot x ) ) \\ &= 0 \cdot x + 2 \cdot ( 1 \cdot x + 6 \cdot x ) \\ &= 0 \cdot x + 2 \cdot ( 7 \cdot x ) \\ &= 0 \cdot x + 14 \cdot x \\ &= 14 \cdot x \end{aligned}$$

    via application of **Horner's rule**.

- Idea: for $b = 2$ we now know

$$r = y \times x = \left( \sum_{i=0}^{n-1} y_i \cdot 2^i \right) \times x = \sum_{i=0}^{n-1} y_i \cdot x \cdot 2^i,$$

plus

- for any $t$,

$$y_i \cdot t = \begin{cases} 0 & \text{if } y_i = 0 \\ t & \text{if } y_i = 1 \end{cases}$$

- for any $t$,

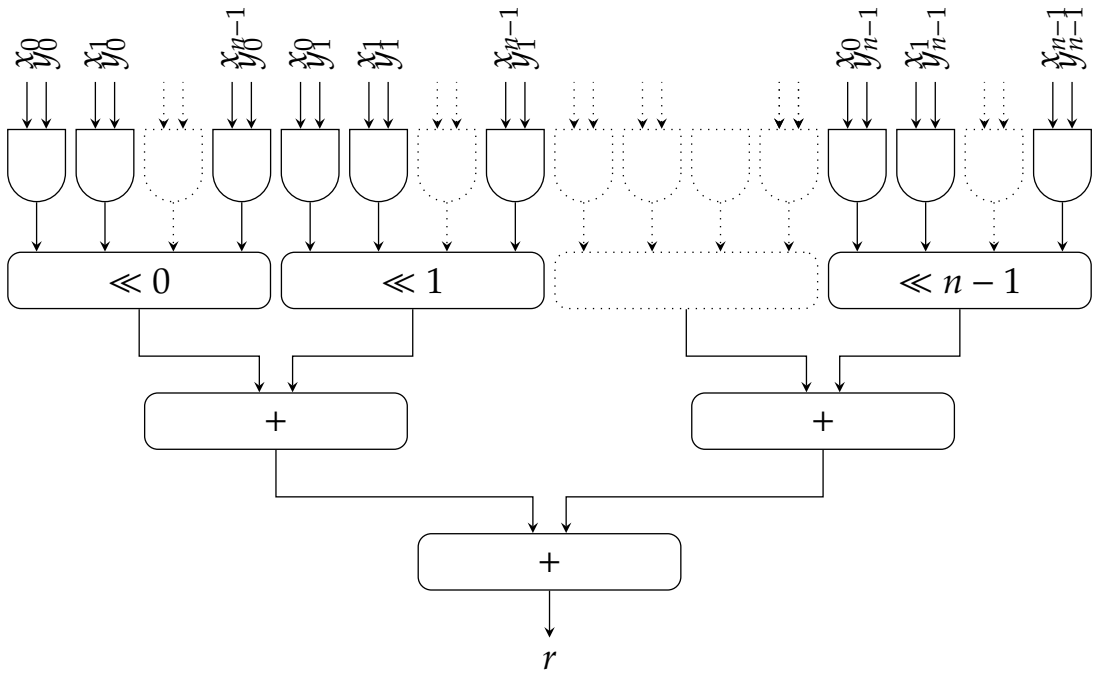
$$t \cdot 2^i \equiv t \ll i,$$

so we can compute $r$ via

1. some AND gates to generate partial products (i.e., $y_i \cdot x$),
2. some left-shift components to scale the partial products correctly (i.e., $y_i \cdot x \cdot 2^i$), and
3. some adder components to sum the scaled partial products.

- Design:

## Circuit




- Evaluation:
  - −ve: requires a larger data-path
  - +ve: requires a smaller control-path (i.e., none at all)
  - +ve: requires less steps (i.e., 1)
  - −ve: has a longer critical path (meaning each step is longer)

- Idea: for $b = 2$ we now know

$$r = y \times x = \left( \sum_{i=0}^{n-1} y_i \cdot 2^i \right) \times x = \sum_{i=0}^{n-1} y_i \cdot x \cdot 2^i,$$

so we can compute $r$ by evaluating the Horner expansion: we

- start with the general step

$$r' \leftarrow y_i \cdot x + 2 \cdot r,$$

- specialise it to read

$$r' \leftarrow \begin{cases} 2 \cdot r & \equiv & (r \ll 1) & \text{if } y_i = 0 \\ x + 2 \cdot r & \equiv & x + (r \ll 1) & \text{if } y_i = 1 \end{cases}$$

- using it to accumuate the result step-by-step.

# Part 3: multiplication in practice: a circuit (3)

## An iterative, bit-serial design

- Idea:

### Algorithm

**Input:** Two unsigned, $n$-bit, base-2 integers $x$ and $y$
**Output:** An unsigned, $2n$-bit, base-2 integer $r = y \cdot x$

```
1 r ← 0
2 for i = n − 1 downto 0 step −1 do
3 |   r ← 2 · r
4 |   if y_i = 1 then
5 |   |   r ← r + x
6 |   end
7 end
8 return r
```

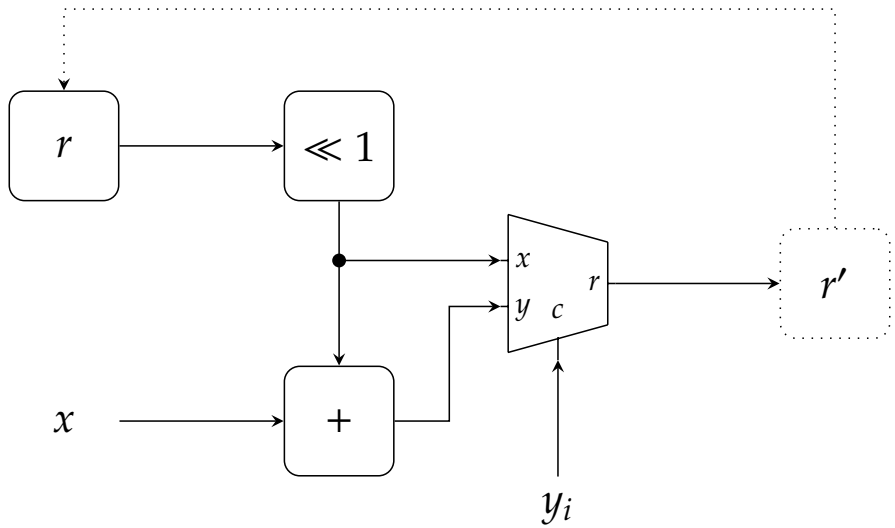
### Example

Consider a case where $y = 14_{(10)} \mapsto 1110_{(2)}$:

| $i$ | $r$ | $y_i$ | $r'$ | |
|---|---|---|---|---|
| | 0 | | | |
| 3 | 0 | 1 | $x$ | $r' \leftarrow 2 \cdot r + x$ |
| 2 | $x$ | 1 | $3 \cdot x$ | $r' \leftarrow 2 \cdot r + x$ |
| 1 | $3 \cdot x$ | 1 | $7 \cdot x$ | $r' \leftarrow 2 \cdot r + x$ |
| 0 | $7 \cdot x$ | 0 | $14 \cdot x$ | $r' \leftarrow 2 \cdot r$ |
| | $14 \cdot x$ | | | |

- Design:

## Circuit




- Evaluation:
  - +ve: requires a smaller data-path
  - −ve: requires a larger control-path (i.e., an entire FSM),
  - −ve: requires more steps (i.e., $n$)
  - +ve: has a shorter critical path (meaning each step is shorter)

# Conclusions

- Take away points:
  1. Computer arithmetic is a broad, interesting (sub-)field:
     - it's a broad topic with a rich history,
     - there's usually a large design space of potential approaches,
     - they're often easy to understand at an intuitive, high level,
     - correctness and efficiency of resulting low-level solutions is vital and challenging.
  2. The strategy we've employed is important and (fairly) general-purpose:
     - explore and understand an approach in theory,
     - translate, formalise, and generalise the approach into an algorithm,
     - translate the algorithm, e.g., into circuit,
     - refine (or select) the circuit to satisfy any design constraints.

# Additional Reading


- *Wikipedia: Computer Arithmetic*. URL: https://en.wikipedia.org/wiki/Category:Computer_arithmetic.
- D. Page. "Chapter 7: Arithmetic and logic". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009.
- B. Parhami. "Part 3: Multiplication". In: *Computer Arithmetic: Algorithms and Hardware Designs*. 1st ed. Oxford University Press, 2000.
- W. Stallings. "Chapter 10: Computer arithmetic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013.
- A.S. Tanenbaum and T. Austin. "Section 3.2.2: Arithmetic circuits". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012.

# References


[1] *Wikipedia: Computer Arithmetic*. URL: https://en.wikipedia.org/wiki/Category:Computer_arithmetic (see p. 19).

[2] D. Page. "Chapter 7: Arithmetic and logic". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009 (see p. 19).

[3] B. Parhami. "Part 3: Multiplication". In: *Computer Arithmetic: Algorithms and Hardware Designs*. 1st ed. Oxford University Press, 2000 (see p. 19).

[4] W. Stallings. "Chapter 10: Computer arithmetic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013 (see p. 19).

[5] A.S. Tanenbaum and T. Austin. "Section 3.2.2: Arithmetic circuits". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012 (see p. 19).