Computer Architecture

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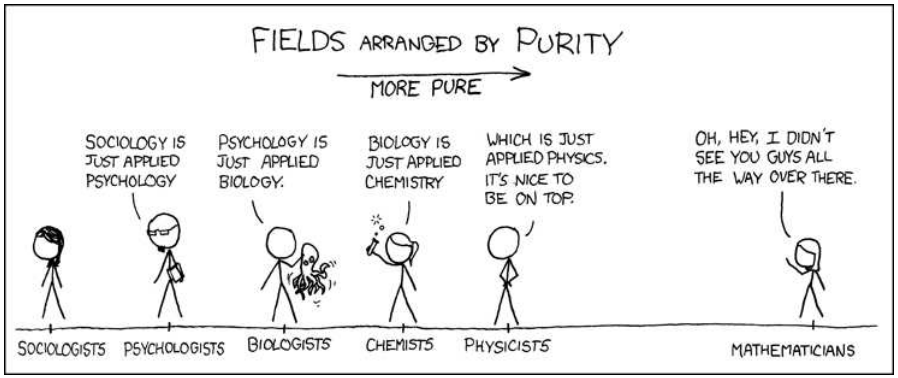
Keep in mind there are *two* PDFs available (of which this is the latter):

1. a PDF of examinable material used as lecture slides, and
2. a PDF of non-examinable, extra material:
   - the associated notes page may be pre-populated with extra, written explaination of material covered in lecture(s), plus
   - anything with a "grey'ed out" header/footer represents extra material which is useful and/or interesting but out of scope (and hence not covered).

Notes:

Notes:



https://xkcd.com/435

Notes:

COMS10015 lecture: week #1

Notes:

- Agenda: an introduction to
  1. **propositional logic**,
  2. **Boolean algebra**, and
  3. application of, i.e., use-cases and rationale for the above within the context of COMS10015.

Part 1: propositional logic (1)

- Concept: a **proposition** is basically a statement

  the temperature is 20°C

  this statement is false
  the temperature is too hot

  which adheres to some simple rules.

Notes:

Part 1: propositional logic (1)

- Concept: a **proposition** is basically a statement

  the temperature is 20°C

  ~~this statement is false~~
  the temperature is too hot

  which adheres to some simple rules, i.e., whose meaning
  1. can be **evaluated** to produce a **truth value**, i.e., **false** or **true**.

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Notes:

Part 1: propositional logic (1)

- Concept: a **proposition** is basically a statement

the temperature is 20°C

~~this statement is false~~
~~the temperature is too hot~~

which adheres to some simple rules, i.e., whose meaning

1. can be **evaluated** to produce a **truth value**, i.e., **false** or **true**,
2. must be unambiguous.

Notes:

Part 1: propositional logic (1)

- Concept: a **proposition** is basically a statement

the temperature is 20°C
the temperature is x°C
~~this statement is false~~
~~the temperature is too hot~~

which adheres to some simple rules, i.e., whose meaning

1. can be **evaluated** to produce a **truth value**, i.e., **false** or **true**,
2. must be unambiguous,
3. can include free **variables**, which must be bound to concrete values before evaluation.

Notes:

Part 1: propositional logic (1)

- Concept: a **proposition** is basically a statement

$$\begin{aligned} f &= \text{the temperature is } 20°C \\ g(x) &= \text{the temperature is } x°C \\ &\quad \sout{\text{this statement is false}} \\ &\quad \sout{\text{the temperature is too hot}} \end{aligned}$$

which adheres to some simple rules, i.e., whose meaning

1. can be **evaluated** to produce a **truth value**, i.e., **false** or **true**,
2. must be unambiguous,
3. can include free **variables**, which must be bound to concrete values before evaluation, and
4. can be represented using a short-hand variable or function.

Notes:

Part 1: propositional logic (2)

- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.
  - single statements can be combined using various **connectives**, e.g.,

$$\text{the temperature is not } 20°C$$

  - so that, adding parentheses where needed to add clarity, we get
    1. "not x" is denoted $\neg x$,

Notes:

Part 1: propositional logic (2)

- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.
  - single statements can be combined using various **connectives**, e.g.,

    $$\neg(\text{the temperature is } 20°C)$$

  - so that, adding parentheses where needed to add clarity, we get
    1. "not x" is denoted $\neg x$,

Notes:

Part 1: propositional logic (2)

- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.
  - single statements can be combined using various **connectives**, e.g.,

    $$\text{the temperature is } 20°C \text{ and it is sunny}$$

  - so that, adding parentheses where needed to add clarity, we get
    1. "not x" is denoted $\neg x$,
    2. "x and y" is denoted $x \wedge y$,

Notes:

Part 1: propositional logic (2)

- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.
  - single statements can be combined using various **connectives**, e.g.,

    $$(\text{the temperature is } 20°C) \wedge (\text{it is sunny})$$

  - so that, adding parentheses where needed to add clarity, we get
    1. "not x" is denoted $\neg x$,
    2. "x and y" is denoted $x \wedge y$,

Notes:

Part 1: propositional logic (2)

- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.
  - single statements can be combined using various **connectives**, e.g.,

    the temperature is $20°C$ or it is sunny

  - so that, adding parentheses where needed to add clarity, we get
    1. "not x" is denoted $\neg x$,
    2. "x and y" is denoted $x \wedge y$,
    3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,

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Notes:

Part 1: propositional logic (2)

- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.
  - single statements can be combined using various **connectives**, e.g.,

    (the temperature is 20°C) ∨ (it is sunny)
  - so that, adding parentheses where needed to add clarity, we get
    1. "not x" is denoted $\neg x$,
    2. "x and y" is denoted $x \wedge y$,
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Notes:

Part 1: propositional logic (2)

- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.
  - single statements can be combined using various **connectives**, e.g.,

    either the temperature is 20°C or it is sunny, but not both

  - so that, adding parentheses where needed to add clarity, we get
    1. "not x" is denoted $\neg x$,
    2. "x and y" is denoted $x \wedge y$,
    3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
    4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,

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Part 1: propositional logic (2)

- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.
  - single statements can be combined using various **connectives**, e.g.,

    $$(\text{the temperature is } 20°C) \oplus (\text{it is sunny})$$

  - so that, adding parentheses where needed to add clarity, we get
    1. "not x" is denoted $\neg x$,
    2. "x and y" is denoted $x \wedge y$,
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    4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,

Notes:

Part 1: propositional logic (2)

- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.
  - single statements can be combined using various **connectives**, e.g.,

    the temperature being $20°C$ implies that it is sunny

  - so that, adding parentheses where needed to add clarity, we get
    1. "not x" is denoted $\neg x$,
    2. "x and y" is denoted $x \wedge y$,
    3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
    4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,
    5. "x implies y" is denoted $x \Rightarrow y$, and sometimes written "if x then y", and

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- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.
  - single statements can be combined using various **connectives**, e.g.,

    $$(\text{the temperature is } 20°C) \Rightarrow (\text{it is sunny})$$

  - so that, adding parentheses where needed to add clarity, we get
    1. "not x" is denoted $\neg x$,
    2. "x and y" is denoted $x \wedge y$,
    3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
    4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,
    5. "x implies y" is denoted $x \Rightarrow y$, and sometimes written "if x then y", and

Notes:

Part 1: propositional logic (2)

- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.
  - single statements can be combined using various **connectives**, e.g.,

    the temperature is $20°C$ is equivalent to it being sunny

  - so that, adding parentheses where needed to add clarity, we get
    1. "not x" is denoted $\neg x$,
    2. "x and y" is denoted $x \wedge y$,
    3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
    4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,
    5. "x implies y" is denoted $x \Rightarrow y$, and sometimes written "if x then y", and
    6. "x is equivalent to y" is denoted $x \equiv y$, and sometimes written "x if and only if y" or "x iff. y".

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- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.
  - single statements can be combined using various **connectives**, e.g.,

    $$(\text{the temperature is } 20^\circ C) \equiv (\text{it is sunny})$$

  - so that, adding parentheses where needed to add clarity, we get
    1. "not x" is denoted $\neg x$,
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Notes:

Part 1: propositional logic (2)

- Question: how can such statements be *represented*?
- Answer #1: using a *expression*-style form.

  - you *might* see more formal terms or different notation for the *same* connectives:
    - $\neg$ is often termed logical **compliment** (or **negation**),
    - $\wedge$ is often termed logical **conjunction**,
    - $\vee$ is often termed logical (inclusive) **disjunction**,
    - $\oplus$ is often termed logical (exclusive) **disjunction**,
    - $\Rightarrow$ is often termed logical **implication**, and
    - $\equiv$ is often termed logical **equivalence**.

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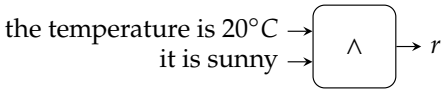
Part 1: propositional logic (3)

- Question: how can such statements be *represented*?
- Answer #2: using a *diagrammatic*-style form, e.g.,

$$r \;=\; (\text{the temperature is } 20°C) \wedge (\text{it is sunny})$$

$$\equiv$$



which, more formally, could be termed a **data-flow diagram**.

Notes:

Part 1: propositional logic (4)

- Question: how can such statements be *evaluated*?
- Answer: in the general case,
  - we know a given statement will evaluate to a truth value, i.e., **false** or **true**,
  - we may want to evaluate for *one*, but can enumerate *all* input combinations,
  - the result is termed a **truth table**, e.g., if

$$\begin{array}{lcl} x & = & \text{the temperature is } 20°C \\ y & = & \text{it is sunny} \\ r & = & (\text{the temperature is } 20°C) \wedge (\text{it is sunny}) \end{array}$$

then

inputs		output
x	y	r
false	**false**	**false**
false	**true**	**false**
true	**false**	**false**
true	**true**	**true**

noting that

1. each row details the output(s) associated with a given assignment to the inputs,
2. if there are n inputs, the truth table will have 2^n rows,
3. one can view the outcome as a form of specification.

Notes:

- Question: how can such statements be *evaluated*?
- Answer: in some specific cases, e.g., those for connectives such as

x	y	$\neg x$	$x \wedge y$	$x \vee y$	$x \oplus y$	$x \Rightarrow y$	$x \equiv y$
false	**false**	**true**	**false**	**false**	**false**	**true**	**true**
false	**true**	**true**	**false**	**true**	**true**	**true**	**false**
true	**false**	**false**	**false**	**true**	**true**	**false**	**false**
true	**true**	**false**	**true**	**true**	**false**	**true**	**true**

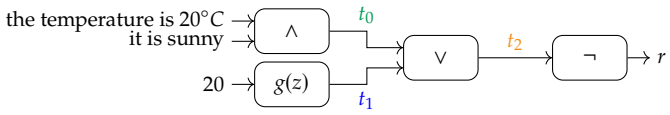
we already know what the associated truth table is.

Notes:

- Question: how can such statements be *evaluated*?
- Example:
  - imagine we have

$$\begin{array}{lcl} x & = & \text{the temperature is } 20^\circ C \\ y & = & \text{it is sunny} \\ g(z) & = & \text{the temperature is } z^\circ C \\ r & = & \neg(((\text{the temperature is } 20^\circ C) \wedge (\text{it is sunny})) \vee (\text{the temperature is } z^\circ C)) \end{array}$$

  which could be translated into the diagrammatic form



  - an example evaluation might be as follows:

inputs		intermediates			output
x	y	t_0	t_1	t_2	r
false	**false**	**false**	**false**	**false**	**true**
false	**true**	**false**	**false**	**false**	**true**
true	**false**	**false**	**true**	**true**	**false**
true	**true**	**true**	**true**	**true**	**false**

Notes:

Part 2: Boolean algebra (1)

- Concept:

1. in **elementary algebra**, for some number x we have that
$$x + 0 = x$$
and
$$x \cdot 1 = x,$$
2. in **set theory**, for some set x we have that
$$x \cup \emptyset = x$$
and
$$x \cap \mathcal{U} = x,$$
3. in **propositional logic**, for some truth value x we have that
$$x \vee \textbf{false} = x$$
and
$$x \wedge \textbf{true} = x.$$

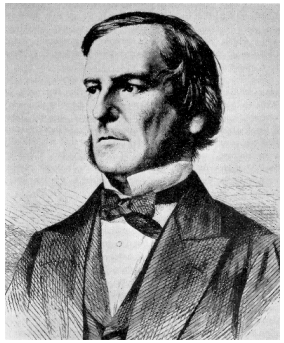
Notes:

Part 2: Boolean algebra (2)

Thou must

1. work with the set $\mathbb{B} = \{0, 1\}$ of **binary** digits, using 0 and 1 instead of **false** and **true**,
2. shorten every statement into either a **variable** *or* **function**,
3. use **unary operators**, e.g., $\neg$ (or NOT), and **binary operators**, e.g., $\wedge$ and $\vee$ (or AND and OR), to form **expressions**,
4. manipulate said expressions according to some axioms (or rules),

then call the result **Boolean algebra**.



https://en.wikipedia.org/wiki/File:George_Boole.jpg

Notes:

- Concept: put more concretely, we now have
  1. a set of operators specified by

Definition

x	y	$\neg x$	$x \wedge y$	$x \vee y$	$x \oplus y$	$x \Rightarrow y$	$x \equiv y$
0	0	1	0	0	0	1	1
0	1	1	0	1	1	1	0
1	0	0	0	1	1	0	0
1	1	0	1	1	0	1	1

Notes:

Part 2: Boolean algebra (3)

- Concept: put more concretely, we now have

  2. a set of axioms that allow manipulation of expressions comprised of said operators, i.e.,

Definition

Name	Axiom(s)			Name	Axiom(s)		
commutativity	$x \wedge y$	$\equiv$	$y \wedge x$	commutativity	$x \vee y$	$\equiv$	$y \vee x$
association	$(x \wedge y) \wedge z$	$\equiv$	$x \wedge (y \wedge z)$	association	$(x \vee y) \vee z$	$\equiv$	$x \vee (y \vee z)$
distribution	$x \wedge (y \vee z)$	$\equiv$	$(x \wedge y) \vee (x \wedge z)$	distribution	$x \vee (y \wedge z)$	$\equiv$	$(x \vee y) \wedge (x \vee z)$

plus other rules such as **precedence** (to deal with ambiguity in the absence of parentheses).

Notes:

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- Concept: put more concretely, we now have

2. a set of axioms that allow manipulation of expressions comprised of said operators, i.e.,

Definition

Name	Axiom(s)	Name	Axiom(s)
identity	$x \wedge 1 \equiv x$	identity	$x \vee 0 \equiv x$
null	$x \wedge 0 \equiv 0$	null	$x \vee 1 \equiv 1$
idempotency	$x \wedge x \equiv x$	idempotency	$x \vee x \equiv x$
inverse	$x \wedge \neg x \equiv 0$	inverse	$x \vee \neg x \equiv 1$

plus other rules such as **precedence** (to deal with ambiguity in the absence of parentheses).

Notes:

Part 2: Boolean algebra (3)

- Concept: put more concretely, we now have

2. a set of axioms that allow manipulation of expressions comprised of said operators, i.e.,

Definition

Name	Axiom(s)	Name	Axiom(s)
absorption	$x \wedge (x \vee y) \equiv x$	absorption	$x \vee (x \wedge y) \equiv x$
de Morgan	$\neg(x \wedge y) \equiv \neg x \vee \neg y$	de Morgan	$\neg(x \vee y) \equiv \neg x \wedge \neg y$

plus other rules such as **precedence** (to deal with ambiguity in the absence of parentheses).

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Notes:

- Concept: put more concretely, we now have

2. a set of axioms that allow manipulation of expressions comprised of said operators, i.e.,

Definition

Name	Axiom(s)
equivalence	$x \equiv y \equiv (x \Rightarrow y) \wedge (y \Rightarrow x)$
implication	$x \Rightarrow y \equiv \neg x \vee y$
involution	$\neg\neg x \equiv x$

plus other rules such as **precedence** (to deal with ambiguity in the absence of parentheses).

Notes:

Definition

The fact there are AND and OR forms of most axioms hints at a more general underlying principle. Consider a Boolean expression e: the **principle of duality** states that the **dual expression** e^D is formed by

1. leaving each variable as is,
2. swapping each $\wedge$ with $\vee$ and vice versa, and
3. swapping each 0 with 1 and vice versa.

Of course e and e^D are different expressions, and clearly not equivalent; if we start with some $e \equiv f$ however, then we do still get $e^D \equiv f^D$.

Example

As an example, consider axioms for

1. distribution, e.g., if
$$e = x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$$
then
$$e^D = x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$$
and
2. identity, e.g., if
$$e = x \wedge 1 \equiv x$$
then
$$e^D = x \vee 0 \equiv x.$$

Notes:

Definition

The de Morgan axiom can be turned into a more general principle. Consider a Boolean expression e: the **principle of complements** states that the **complement expression** $\neg e$ is formed by

1. swapping each variable x with the complement $\neg x$,
2. swapping each $\wedge$ with $\vee$ and vice versa, and
3. swapping each 0 with 1 and vice versa.

Example

As an example, consider that if

$$e = x \wedge y \wedge z,$$

then by the above we should find

$$f = \neg e = (\neg x) \vee (\neg y) \vee (\neg z).$$

Proof:

x	y	z	$\neg x$	$\neg y$	$\neg z$	e	f
0	0	0	1	1	1	0	1
0	0	1	1	1	0	0	1
0	1	0	1	0	1	0	1
0	1	1	1	0	0	0	1
1	0	0	0	1	1	0	1
1	0	1	0	1	0	0	1
1	1	0	0	0	1	0	1
1	1	1	0	0	0	1	0

Notes:

- Concept: we can define various **derived operators** in terms of NOT, AND, and OR.
- Example:
  - "exclusive-OR" or **XOR**, such that

$$x \oplus y \equiv (\neg x \wedge y) \vee (x \wedge \neg y)$$

so

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

Notes:

Part 2: Boolean algebra (6)

- Concept: we can define various **derived operators** in terms of NOT, AND, and OR.
- Example:
  - "NOT-AND" or **NAND**, such that

$$x \overline{\wedge} y \equiv \neg(x \wedge y)$$

so

x	y	$x \overline{\wedge} y$
0	0	1
0	1	1
1	0	1
1	1	0

  - "NOT-OR" or **NOR**, such that

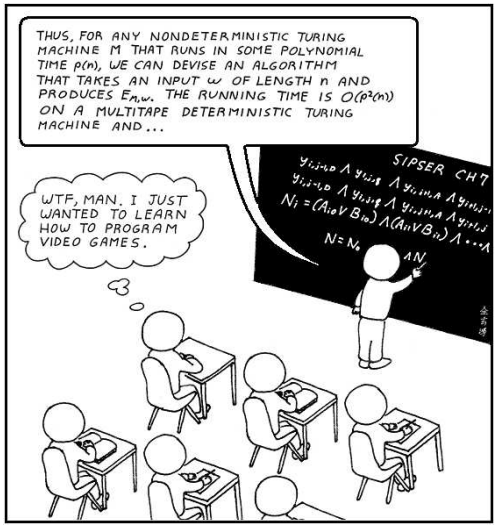
$$x \overline{\vee} y \equiv \neg(x \vee y)$$

so

x	y	$x \overline{\vee} y$
0	0	1
0	1	0
1	0	0
1	1	0

Notes:

Part 3: towards practical use-cases (1)



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Notes:

- Concept:
  - Boolean algebra plays a role in *many* contexts, e.g.,

r is x	$\equiv$	$r = x$	$\cong$	`r == x`	$\cong$	`r = x`
r is NOT x	$\equiv$	$r = \neg x$	$\cong$	`r != x`	$\cong$	`r = ~x`
r is x NAND y	$\equiv$	$r = x \overline{\wedge} y$	$\cong$	`r != x && y`	$\cong$	`r = ~( x & y )`
r is x NOR y	$\equiv$	$r = x \overline{\vee} y$	$\cong$	`r != x || y`	$\cong$	`r = ~( x | y )`
r is x AND y	$\equiv$	$r = x \wedge y$	$\cong$	`r == x && y`	$\cong$	`r = x & y`
r is x OR y	$\equiv$	$r = x \vee y$	$\cong$	`r == x || y`	$\cong$	`r = x | y`
r is x XOR y	$\equiv$	$r = x \oplus y$			$\cong$	`r = x ^ y`

  - note that the latter columns capture
    - **decisional** operators, e.g., && : $\mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$, and
    - **computational** operators, e.g., & : $\mathbb{B}^n \times \mathbb{B}^n \rightarrow \mathbb{B}^n$.

Notes:

Listing

```
1  void condition_v1( int a, int b, int c, int d ) {
2    if( ( ( a + b ) > c ) && ( ( d % 2 ) == 0 ) ) {
3      //         ( a + b is    >  c ) and     ( d is    even )
4
5      printf( "if -> true  : %d %d %d %d\n", a, b, c, d );
6    }
7    else {
8      //   not ( ( a + b is    >  c ) and     ( d is    even ) )
9      //
10     // = not   ( a + b is    >  c ) or  not ( d is    even )
11     // =       ( a + b isn't >  c ) or      ( d isn't even )
12     // =       ( a + b is    <= c ) or      ( d is    odd  )
13
14     printf( "if -> false : %d %d %d %d\n", a, b, c, d );
15   }
16 }
```


Notes:

Listing

```
1 void condition_v2( int a, int b, int c, int d ) {
2   if( ( ( a + b ) > c ) && ( d = 3 ) ) {
3     printf( "if -> true  : %d %d %d %d\n", a, b, c, d );
4   }
5   else {
6     printf( "if -> false : %d %d %d %d\n", a, b, c, d );
7   }
8 }
```

Notes:

- Question: simplify the Boolean expression

$$f = (\neg(a \vee b) \wedge \neg(c \vee d \vee e)) \vee \neg(a \vee b)$$

into a form that contains the fewest operators possible.

Notes:

Part 3: a (more) practical perspective (4)
Formalisation ⇝ optimisation

- Question: simplify the Boolean expression

$$f = (\neg(a \vee b) \wedge \neg(c \vee d \vee e)) \vee \neg(a \vee b)$$

into a form that contains the fewest operators possible.

- Solution #1: less steps.

$$\begin{array}{lcccccll} f & = & (\neg(a \vee b) & \wedge & \neg(c \vee d \vee e)) & \vee & \neg(a \vee b) & \\ & = & \neg(a \vee b) & \vee & (\neg(a \vee b) & \wedge & \neg(c \vee d \vee e)) & \text{(commutativity)} \\ & = & \neg(a \vee b) & & & & & \text{(absorption)} \end{array}$$

Notes:

Part 3: a (more) practical perspective (4)
Formalisation ⇝ optimisation

- Question: simplify the Boolean expression

$$f = (\neg(a \vee b) \wedge \neg(c \vee d \vee e)) \vee \neg(a \vee b)$$

into a form that contains the fewest operators possible.

- Solution #2: more steps.

$$\begin{array}{lcccccll} f & = & (\neg(a \vee b) & \wedge & \neg(c \vee d \vee e)) & \vee & \neg(a \vee b) & \\ & = & ((\neg a \wedge \neg b) & \wedge & \neg(c \vee d \vee e)) & \vee & \neg(a \vee b) & \text{(de Morgan)} \\ & = & ((\neg a \wedge \neg b) & \wedge & (\neg c \wedge \neg d \wedge \neg e)) & \vee & \neg(a \vee b) & \text{(de Morgan)} \\ & = & ((\neg a \wedge \neg b) & \wedge & (\neg c \wedge \neg d \wedge \neg e)) & \vee & (\neg a \wedge \neg b) & \text{(de Morgan)} \\ & = & (\neg a \wedge \neg b) & \vee & ((\neg a \wedge \neg b) & \wedge & (\neg c \wedge \neg d \wedge \neg e)) & \text{(commutativity)} \\ & = & (\neg a \wedge \neg b) & & & & & \text{(absorption)} \\ & = & \neg(a \vee b) & & & & & \text{(de Morgan)} \end{array}$$

Notes:

Part 3: a (more) practical perspective (5)
Formalisation ⇝ optimisation

- Question: simplify the Boolean expression

$$g = (a \wedge b \wedge c) \vee (\neg a \wedge b) \vee (a \wedge b \wedge \neg c)$$

into a form that contains the fewest operators possible.

Notes:

Part 3: a (more) practical perspective (5)
Formalisation ⇝ optimisation

- Question: simplify the Boolean expression

$$g = (a \wedge b \wedge c) \vee (\neg a \wedge b) \vee (a \wedge b \wedge \neg c)$$

into a form that contains the fewest operators possible.

- Solution:

$$\begin{array}{lllllll}
g & = & (a \wedge b \wedge c) & \vee & (\neg a \wedge b) & \vee & (a \wedge b \wedge \neg c) \\
& = & (a \wedge b \wedge c) & \vee & (a \wedge b \wedge \neg c) & \vee & (\neg a \wedge b) \quad \text{(commutativity)} \\
& = & (a \wedge b) & \wedge & (c \vee \neg c) & \vee & (\neg a \wedge b) \quad \text{(distribution)} \\
& = & (a \wedge b) & \wedge & 1 & \vee & (\neg a \wedge b) \quad \text{(inverse)} \\
& = & (a \wedge b) & & & \vee & (\neg a \wedge b) \quad \text{(identity)} \\
& = & b & \wedge & (a \vee \neg a) & & \quad \text{(distribution)} \\
& = & b & \wedge & 1 & & \quad \text{(inverse)} \\
& = & b & & & & \quad \text{(identity)}
\end{array}$$

Notes:

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- Question: simplify the Boolean expression

$$g = (a \wedge b \wedge c) \vee (\neg a \wedge b) \vee (a \wedge b \wedge \neg c)$$

into a form that contains the fewest operators possible.

- Proof:

a	b	c	$\neg a$	$\neg b$	$\neg c$	t_0	t_1	t_2	g
0	0	0	1	1	1	0	0	0	0
0	0	1	1	1	0	0	0	0	0
0	1	0	1	0	1	0	1	0	1
0	1	1	1	0	0	0	1	0	1
1	0	0	0	1	1	0	0	0	0
1	0	1	0	1	0	0	0	0	0
1	1	0	0	0	1	0	0	1	1
1	1	1	0	0	0	1	0	0	1

where

$$\begin{aligned} t_0 &= a \wedge b \wedge c \\ t_1 &= \neg a \wedge b \\ t_2 &= a \wedge b \wedge \neg c \end{aligned}$$

such that $g = t_0 \vee t_1 \vee t_2$.

Notes:

Quote

If I designed a computer with 200 *chips, I tried to design it with* 150*. And then I would try to design it with* 100*. I just tried to find every trick I could in life to design things real tiny.*

– Wozniak

Quote

So I took 20 *chips off their board; I bypassed* 20 *of their chips.*

– Wozniak

Notes:

- The quotes relate to design and implementation of a (floppy) disk controller for the Apple II computer (circa 1977); there is an obvious focus on efficiency, which is credited as allowing the controller to be commercially viable. A detailed overview of the overarching anecdote is available via

  https://en.wikipedia.org/wiki/Disk_II

  or

  https://apple2history.org/history/ah05/

  The moral is that, in reality, "it works", while important, may not be good enough: meeting various other (market-driven) quality metrics (e.g., efficiency, physical size, power consumption, etc.) is often vital rather than simply attractive.

https://en.wikipedia.org/wiki/File:Shugart_SA400.jpg
https://en.wikipedia.org/wiki/File:Interface_Card_-_Disk_II_Interface_Apple2.jpg

Notes:

Definition

Consider a Boolean expression:

1. When the expression is written as a sum (i.e., OR) of terms which each comprise the product (i.e., AND) of variables, e.g.,
$$\underbrace{(a \wedge b \wedge c)}_{\text{minterm}} \vee (d \wedge e \wedge f),$$
it is said to be in **disjunctive normal form** or **Sum of Products (SoP)** form; the terms are called the **minterms**. Note that each variable can exist as-is *or* complemented using NOT, meaning
$$\underbrace{(\neg a \wedge b \wedge c)}_{\text{minterm}} \vee (d \wedge \neg e \wedge f),$$
is also a valid SoP expression.
2. When the expression is written as a product (i.e., AND) of terms which each comprise the sum (i.e., OR) of variables, e.g.,
$$\underbrace{(a \vee b \vee c)}_{\text{maxterm}} \wedge (d \vee e \vee f),$$
it is said to be in **conjunctive normal form** or **Product of Sums (PoS)** form; the terms are called the **maxterms**. As above each variable can exist as-is *or* complemented using NOT.

Notes:

- Concept: **Electronic Design Automation (EDA)** [3], applied to tasks such as
  - manipulation,
  - translation,
  - simulation,
  - verification,
  - ...

  of functionality related to and explained by Boolean algebra.

Notes:

- Seymour Cray (a *super*computer architect) produced some interesting anecdotes about use of Computer-Aided Design (CAD) tools, which one could view as a more general case of EDA, in his design process: see for example

  https://en.wikipedia.org/wiki/Seymour_Cray

Listing

```
1  from sympy                import *
2  from sympy.logic.boolalg import simplify_logic
3
4  a, b, c, d, e = symbols( "a b c d e" )
5
6  f = ( ~( a | b ) & ~( c | d | e ) ) | ~( a | b )
7
8  print(                    f   )
9  print( simplify_logic( f ) )
10
11 g = ( a & b & c ) | ( ~a & b ) | ( a & b & ~c )
12
13 print(                    g   )
14 print( simplify_logic( g ) )
```

Output

```
1 ~(a | b) | (~(a | b) & ~(c | d | e))
2 ~a & ~b
3 (b & ~a) | (a & b & c) | (a & b & ~c)
4 b
```

Notes:

- Note that this example can be explored online via

  https://live.sympy.org

- Question: imagine we have a Boolean expression (or circuit)

$$f(x_0, x_1, \ldots x_{n-1})$$

and manipulate it into a different form

$$g(x_0, x_1, \ldots x_{n-1})$$

e.g., optimise it: we want to know if there is a bug in g.

Notes:

Part 3: a (more) practical perspective (11)
Formalisation ⇝ verification

- Question: imagine we have a Boolean expression (or circuit)

$$f(x_0, x_1, \ldots x_{n-1})$$

and manipulate it into a different form

$$g(x_0, x_1, \ldots x_{n-1})$$

e.g., optimise it: we want to know if there is a bug in g.

- Solution(s):
  1. *prove* that f and g are equivalent,
  2. enumerate all 2^n possible input combinations, and see if f ever differs from g, or
  3. use a (carefully defined) instance of **SAT**.

Notes:

Definition

The **Boolean satisfiability problem** (or **SAT**) is a decision problem. Consider some Boolean function

$$f(x_0, x_1, \ldots x_{n-1}).$$

which defines a SAT **instance**. The problem is to decide whether or not an assignment to said variables (i.e., $x_i \in \{0, 1\}$ for $0 \leq i < n$) exists st. f is **satisfiable** (i.e., we have $f(x_0, x_1, \ldots x_{n-1}) = 1$).

Notes:

- SAT is NP-complete [8, 9] meaning that
  1. no known algorithm will efficiently solve every SAT instance, but
  2. other NP problems can be expressed as SAT instances.

- Question: decide whether

$$f(a, b) = (b \wedge \neg a) \vee (a \wedge \neg b)$$

$$\equiv$$

$$g(a, b) = (a \vee b) \wedge \neg(a \wedge b).$$

Notes:

- The idea of the SAT instance is to use how XOR works: what we're saying is that if h is satisfiable, then we have an assignment to the variables st. the result of f and g *differ* (XOR produces 1 as output if one (and only one) of the inputs are 1, and 0 otherwise).
- One underlying idea here is that each approach is valid, but also might be advantageous or disadvantageous when used in a particular context: the use of axiomatic manipulation does not *tell* us anything (e.g., why f and g differ, if they do), for example, and brute-force enumeration will clearly become intractable very quickly (as n grows). The other is that greater understanding of the relationship between theory and practice allows better results in both: here, in a rough sense,
  - in theory, SAT helps us understand *what* we can compute, while
  - in practice, SAT helps us understand *how* we compute it.

- Question: decide whether

$$f(a,b) = (b \wedge \neg a) \vee (a \wedge \neg b)$$

$$\equiv$$

$$g(a,b) = (a \vee b) \wedge \neg(a \wedge b).$$

- Solution #1: manipulation via axioms, e.g.,

$$\begin{aligned}
f(a,b) &= (b \wedge \neg a) && \vee && (a \wedge \neg b) && \\
&= (b \wedge \neg a) \vee 0 && \vee && (a \wedge \neg b) \vee 0 && \text{(identity)} \\
&= (b \wedge \neg a) \vee (b \wedge \neg b) && \vee && (a \wedge \neg b) \vee (a \wedge \neg a) && \text{(inverse)} \\
&= (b \wedge (\neg a \vee \neg b)) && \vee && (a \wedge (\neg b \vee \neg a)) && \text{(distribution)} \\
&= ((\neg a \vee \neg b) \wedge b) && \vee && ((\neg a \vee \neg b) \wedge a) && \text{(commutativity)} \\
&= (\neg a \vee \neg b) \wedge (a \vee b) && && && \text{(distribution)} \\
&= (a \vee b) \wedge (\neg a \vee \neg b) && && && \text{(commutativity)} \\
&= (a \vee b) \wedge \neg(a \wedge b) && && && \text{(de Morgan)} \\
&= g(a,b) && && &&
\end{aligned}$$

Notes:

- The idea of the SAT instance is to use how XOR works: what we're saying is that if h is satisfiable, then we have an assignment to the variables st. the result of f and g *differ* (XOR produces 1 as output if one (and only one) of the inputs are 1, and 0 otherwise).
- One underlying idea here is that each approach is valid, but also might be advantageous or disadvantageous when used in a particular context: the use of axiomatic manipulation does not *tell* us anything (e.g., why f and g differ, if they do), for example, and brute-force enumeration will clearly become intractable very quickly (as n grows). The other is that greater understanding of the relationship between theory and practice allows better results in both: here, in a rough sense,
  - in theory, SAT helps us understand *what* we can compute, while
  - in practice, SAT helps us understand *how* we compute it.

Part 3: a (more) practical perspective (13)
Formalisation ⇝ verification

- Question: decide whether

$$f(a,b) = (b \wedge \neg a) \vee (a \wedge \neg b)$$

$$\equiv$$

$$g(a,b) = (a \vee b) \wedge \neg(a \wedge b).$$

- Solution #2: brute-force enumeration, i.e.,

a	b	$b \wedge \neg a$	$a \wedge \neg b$	$f(a,b)$	$a \vee b$	$\neg(a \wedge b)$	$g(a,b)$
0	0	0	0	0	0	1	0
0	1	1	0	1	1	1	1
1	0	0	1	1	1	1	1
1	1	0	0	0	1	0	0

Notes:

- The idea of the SAT instance is to use how XOR works: what we're saying is that if h is satisfiable, then we have an assignment to the variables st. the result of f and g *differ* (XOR produces 1 as output if one (and only one) of the inputs are 1, and 0 otherwise).
- One underlying idea here is that each approach is valid, but also might be advantageous or disadvantageous when used in a particular context: the use of axiomatic manipulation does not *tell* us anything (e.g., why f and g differ, if they do), for example, and brute-force enumeration will clearly become intractable very quickly (as n grows). The other is that greater understanding of the relationship between theory and practice allows better results in both: here, in a rough sense,
  - in theory, SAT helps us understand *what* we can compute, while
  - in practice, SAT helps us understand *how* we compute it.

- Question: decide whether

$$f(a,b) = (b \wedge \neg a) \vee (a \wedge \neg b)$$

$$\equiv$$

$$g(a,b) = (a \vee b) \wedge \neg(a \wedge b).$$

- Solution #3: define

$$h(x_0, x_1, \ldots x_{n-1}) = f(x_0, x_1, \ldots x_{n-1}) \oplus g(x_0, x_1, \ldots x_{n-1})$$

and use this as a SAT instance: if the SAT instance is satisfiable, this yields a **test vector** we can use to debug g.

Notes:

- The idea of the SAT instance is to use how XOR works: what we're saying is that if h is satisfiable, then we have an assignment to the variables st. the result of f and g *differ* (XOR produces 1 as output if one (and only one) of the inputs are 1, and 0 otherwise).
- One underlying idea here is that each approach is valid, but also might be advantageous or disadvantageous when used in a particular context: the use of axiomatic manipulation does not *tell* us anything (e.g., why f and g differ, if they do), for example, and brute-force enumeration will clearly become intractable very quickly (as n grows). The other is that greater understanding of the relationship between theory and practice allows better results in both: here, in a rough sense,
  - in theory, SAT helps us understand *what* we can compute, while
  - in practice, SAT helps us understand *how* we compute it.

Listing

```
1  from sympy                 import *
2  from sympy.logic.inference import satisfiable
3
4  a, b, c, d, e = symbols( "a b c d e" )
5
6  f = ( b & ~a ) | ( a & ~b )
7  g = ( a | b ) & ~( a & b )
8
9  print(              f       )
10 print(                  g   )
11 print( satisfiable( f ^ g ) )
```

Output

```
1 (a & ~b) | (b & ~a)
2 (a | b) & ~(a & b)
3 False
```

Notes:

- Note that this example can be explored online via
  https://live.sympy.org

- Concept: truth tables can accommodate **don't care** entries, e.g.,

x	y	r
?	0	1
0	1	**?**
1	1	0

such that

- a **?** (rather than 0 or 1) means we "don't care" (≠ "don't know"),
- on the LHS, for an *in*put,
  - **?** is a wildcard (or short-hand),
  - it means 0 *and* 1,
  - we've compressed two truth table rows into one.
- on the RHS, for an *out*put,
  - **?** is a choice,
  - it means 0 *or* 1,
  - we can select which one to, e.g., optimise the associated expression.

Notes:

- Concept: NAND and NOR are **universal** (or **functionally complete**), e.g.,

$$\begin{array}{lcl} \neg x & \equiv & x \mathbin{\overline{\wedge}} x \\ x \wedge y & \equiv & (x \mathbin{\overline{\wedge}} y) \mathbin{\overline{\wedge}} (x \mathbin{\overline{\wedge}} y) \\ x \vee y \equiv \neg x \mathbin{\overline{\wedge}} \neg y & \equiv & (x \mathbin{\overline{\wedge}} x) \mathbin{\overline{\wedge}} (y \mathbin{\overline{\wedge}} y) \end{array}$$

which we can prove via

x	y	$x \mathbin{\overline{\wedge}} y$	$x \mathbin{\overline{\wedge}} x$	$y \mathbin{\overline{\wedge}} y$	$(x \mathbin{\overline{\wedge}} y) \mathbin{\overline{\wedge}} (x \mathbin{\overline{\wedge}} y)$	$(x \mathbin{\overline{\wedge}} x) \mathbin{\overline{\wedge}} (y \mathbin{\overline{\wedge}} y)$
0	0	1	1	1	0	0
0	1	1	1	0	0	1
1	0	1	0	1	0	1
1	1	0	0	0	1	1

∴ *any* Boolean function can be expressed using a *single* operator.

Notes:

- Question: translate

$$x \wedge (y \vee z)$$

into a version using NAND only.

Notes:

Part 3: a (more) practical perspective (17)
Constraints ⇝ utilisation

- Question: translate

$$x \wedge (y \vee z)$$

into a version using NAND only.

- Solution #1: apply the identities *naively* to get

$$\begin{aligned} & x \wedge (y \vee z) \\ = \; & x \wedge ((y \mathbin{\overline{\wedge}} y) \mathbin{\overline{\wedge}} (z \mathbin{\overline{\wedge}} z)) \\ = \; & (x \mathbin{\overline{\wedge}} ((y \mathbin{\overline{\wedge}} y) \mathbin{\overline{\wedge}} (z \mathbin{\overline{\wedge}} z))) \mathbin{\overline{\wedge}} (x \mathbin{\overline{\wedge}} ((y \mathbin{\overline{\wedge}} y) \mathbin{\overline{\wedge}} (z \mathbin{\overline{\wedge}} z))) \end{aligned}$$

Notes:

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- Question: translate

$$x \wedge (y \vee z)$$

into a version using NAND only.

- Solution #2: apply the identities *intelligently* to get

$$\begin{array}{rl} & x \wedge (y \vee z) \\ = & x \wedge ((y \mathbin{\overline{\wedge}} y) \mathbin{\overline{\wedge}} (z \mathbin{\overline{\wedge}} z)) \\ = & t \mathbin{\overline{\wedge}} t \end{array}$$

where $t = x \mathbin{\overline{\wedge}} ((y \mathbin{\overline{\wedge}} y) \mathbin{\overline{\wedge}} (z \mathbin{\overline{\wedge}} z))$ is a common sub-expression [2].

Notes:

Conclusions

- Take away points:
  1. Boolean algebra is a (somewhat) cosmetic extension of what you already know.
  2. The design of computational devices, e.g., micro-processors, *isn't* ad hoc: Boolean algebra offers a theoretical basis for reasoning about computational devices (and computation).
     - *any* Boolean function f which can be specified using a truth table can be computed using an associated Boolean expression,
     - a Boolean expression is composed of Boolean operators,
     - *if* we (physically) implement the Boolean operators, we can implement the Boolean expression and hence compute f.

Notes:

Conclusions

- Take away points:

3. We'll focus on *application* (i.e., *use*) versus *theory* (e.g., *study*) of Boolean algebra from here on.
4. Keep in mind that
   - "theory ≠ practice",
   - "*a* solution" ≠ "*the* solution"; "it works" ≠ "it works *well*",
   - worse-is-better > do-the-right-thing [4] ⟹ "get the job done",
   - ...

Notes:

Additional Reading

- *Wikipedia: Boolean algebra*. URL: https://en.wikipedia.org/wiki/Boolean_algebra.
- D. Page. "Chapter 1: Mathematical preliminaries". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009.
- W. Stallings. "Chapter 11: Digital logic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013.
- A.S. Tanenbaum and T. Austin. "Section 3.1: Gates and Boolean algebra". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012.

Notes:

References

[1] *Wikipedia: Boolean algebra*. URL: https://en.wikipedia.org/wiki/Boolean_algebra (see p. 135).

[2] *Wikipedia: Common sub-expression elimination*. URL: https://en.wikipedia.org/wiki/Common_subexpression_elimination (see pp. 127, 129).

[3] *Wikipedia: Electronic Design Automation (EDA)*. URL: https://en.wikipedia.org/wiki/Electronic_design_automation (see p. 101).

[4] *Wikipedia: Worse is better*. URL: https://en.wikipedia.org/wiki/Worse_is_better (see pp. 131, 133).

[5] D. Page. "Chapter 1: Mathematical preliminaries". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009 (see p. 135).

[6] W. Stallings. "Chapter 11: Digital logic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013 (see p. 135).

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[8] S. Cook. "The complexity of theorem proving procedures". In: *ACM Symposium on Theory of Computing (STOC)*. 1971, pp. 151–158 (see p. 110).

[9] L. Levin. "Universal search problems". In: *Problems of Information Transmission* 9.3 (1973), pp. 265–266 (see p. 110).

Notes: