

Computer Architecture

Daniel Page

Department of Computer Science,
University Of Bristol,
Merchant Venturers Building,
Woodland Road,
Bristol, BS8 1UB. UK.
(csdsp@bristol.ac.uk)

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Keep in mind there are *two* PDFs available (of which this is the latter):

1. a PDF of examinable material used as lecture slides, and
2. a PDF of non-examinable, extra material:
 - ▶ the associated notes page may be pre-populated with extra, written explanation of material covered in lecture(s), plus
 - ▶ anything with a “grey’ed out” header/footer represents extra material which is useful and/or interesting but out of scope (and hence not covered).

Notes:

Notes:

- **Problem:** we have $\neg$, $\wedge$, and $\vee$, so given the specification

x_{n-1}	$\cdots$	x_1	x_0	r
0	$\cdots$	0	0	1
0	$\cdots$	0	1	0
0	$\cdots$	1	0	1
0	$\cdots$	1	1	0
$\vdots$		$\vdots$	$\vdots$	$\vdots$
1	$\cdots$	1	1	0

for some Boolean function

$$r = f(x_0, x_1, \dots, x_{n-1}),$$

design a Boolean expression e which can compute it.

- **Solution:** ?

Notes:

- **Agenda:** **combinatorial logic** design, where, crucially,
- the output is a function of the input only,
 - computation is viewed as being continuous,
- via coverage of
1. special-purpose design patterns,
 2. special-purpose building blocks, and
 3. general-purpose derivation.

Notes:

Part 1: special-purpose design patterns

► Pattern #1: reuse (or sharing).

- Imagine, for example, that we are given a 2-input, 1-bit AND gate.
- If, within some larger circuit, we compute

$$r = x \wedge y$$

and then, somewhere else,

$$r' = x \wedge y$$

then we can replace the two AND gates with one: clearly

$$r = r',$$

so we can share one definition between two usage points.

- Formally, this optimisation is type of **Common Sub-expression Elimination (CSE)**.

Notes:

Part 1: special-purpose design patterns

► Pattern #1: “don’t (re)compute a value you’ve computed already”.

- Imagine, for example, that we are given a 2-input, 1-bit AND gate.
- If, within some larger circuit, we compute

$$r = x \wedge y$$

and then, somewhere else,

$$r' = x \wedge y$$

then we can replace the two AND gates with one: clearly

$$r = r',$$

so we can share one definition between two usage points.

- Formally, this optimisation is type of **Common Sub-expression Elimination (CSE)**.

Notes:

Part 1: special-purpose design patterns

► Pattern #2: decomposition (or divide-and-conquer).

- Given an n -input, m -output Boolean function

$$f : \mathbb{B}^n \rightarrow \mathbb{B}^m$$

we can ensure

$$f(x) \equiv f_0(x) \parallel f_1(x) \parallel \dots \parallel f_{m-1}(x)$$

for appropriate f_i .

- Those f_i are such that

$$\begin{array}{ll} f_0 & : \mathbb{B}^n \rightarrow \mathbb{B} \\ f_1 & : \mathbb{B}^n \rightarrow \mathbb{B} \\ & \vdots \\ f_{m-1} & : \mathbb{B}^n \rightarrow \mathbb{B} \end{array}$$

i.e., f_i uses x to compute the i -th output f would.

Notes:

Part 1: special-purpose design patterns

► Pattern #2: “describe a large, complex function in terms of smaller, simple functions”.

- Given an n -input, m -output Boolean function

$$f : \mathbb{B}^n \rightarrow \mathbb{B}^m$$

we can ensure

$$f(x) \equiv f_0(x) \parallel f_1(x) \parallel \dots \parallel f_{m-1}(x)$$

for appropriate f_i .

- Those f_i are such that

$$\begin{array}{ll} f_0 & : \mathbb{B}^n \rightarrow \mathbb{B} \\ f_1 & : \mathbb{B}^n \rightarrow \mathbb{B} \\ & \vdots \\ f_{m-1} & : \mathbb{B}^n \rightarrow \mathbb{B} \end{array}$$

i.e., f_i uses x to compute the i -th output f would.

Notes:

Part 1: special-purpose design patterns

► Pattern #3: independent replication.

- Imagine, for example, that we are given a 2-input, 1-bit AND gate.
- A 2-input, m -bit AND gate is simply replication of 2-input, 1-bit AND gates, i.e.,

$$r = x \wedge y$$

is computed via

$$r_i = x_i \wedge y_i$$

for $0 \leq i < m$.

- For $n = 4$, as an example, this means

$$\begin{array}{rcl} r_0 & = & x_0 \wedge y_0 \\ r_1 & = & x_1 \wedge y_1 \\ r_2 & = & x_2 \wedge y_2 \\ r_3 & = & x_3 \wedge y_3 \end{array}$$

Notes:

Part 1: special-purpose design patterns

► Pattern #3: “repeat the same structure multiple times; the outcomes are separate”.

- Imagine, for example, that we are given a 2-input, 1-bit AND gate.
- A 2-input, m -bit AND gate is simply replication of 2-input, 1-bit AND gates, i.e.,

$$r = x \wedge y$$

is computed via

$$r_i = x_i \wedge y_i$$

for $0 \leq i < m$.

- For $n = 4$, as an example, this means

$$\begin{array}{rcl} r_0 & = & x_0 \wedge y_0 \\ r_1 & = & x_1 \wedge y_1 \\ r_2 & = & x_2 \wedge y_2 \\ r_3 & = & x_3 \wedge y_3 \end{array}$$

Notes:

Part 1: special-purpose design patterns

► Pattern #4: dependent replication.

- Imagine, for example, that we are given a 2-input, 1-bit AND gate.
- An n -input, 1-bit AND gate is simply replication of 2-input, 1-bit AND gates, i.e.,

$$r = \bigwedge_{i=0}^{n-1} x_i$$

is computed via

$$r = x_0 \wedge (x_1 \wedge \cdots (x_{n-1}))$$

which is often termed a **reduction** with respect to AND.

- For $n = 4$, as an example, this means

$$\begin{aligned} r &= x_0 \wedge (x_1 \wedge x_2 \wedge x_3) \\ &= x_0 \wedge x_1 \wedge x_2 \wedge x_3 \\ &= (x_0 \wedge x_1) \wedge (x_2 \wedge x_3) \end{aligned}$$

Notes:

Part 1: special-purpose design patterns

► Pattern #4: “repeat the same structure multiple times; the outcomes are combined”.

- Imagine, for example, that we are given a 2-input, 1-bit AND gate.
- An n -input, 1-bit AND gate is simply replication of 2-input, 1-bit AND gates, i.e.,

$$r = \bigwedge_{i=0}^{n-1} x_i$$

is computed via

$$r = x_0 \wedge (x_1 \wedge \cdots (x_{n-1}))$$

which is often termed a **reduction** with respect to AND.

- For $n = 4$, as an example, this means

$$\begin{aligned} r &= x_0 \wedge (x_1 \wedge x_2 \wedge x_3) \\ &= x_0 \wedge x_1 \wedge x_2 \wedge x_3 \\ &= (x_0 \wedge x_1) \wedge (x_2 \wedge x_3) \end{aligned}$$

Notes:

Part 2: special-purpose building blocks (1)

Choice

► Concept: selection (or choice).

1. a **multiplexer**
 - has m inputs,
 - has 1 output,
 - uses a $(\lceil \log_2(m) \rceil)$ -bit control signal input to choose which input is connected to the output,
- while
2. a **demultiplexer**
 - has 1 input,
 - has m outputs,
 - uses a $(\lceil \log_2(m) \rceil)$ -bit control signal input to choose which output is connected to the input,

noting that

- the input(s) and output(s) are n -bit, but clearly must match up,
- the connection made is continuous, since both components are combinatorial.

Notes:

Part 2: special-purpose building blocks (2)

Choice

► Concept: by analogy,

1. the C `switch` statement

Listing

```
1 switch( c ) {  
2   case 0 : r = w; break;  
3   case 1 : r = x; break;  
4   case 2 : r = y; break;  
5   case 3 : r = z; break;  
6 }
```

acts similarly to a 4-input multiplexer,

2. the C `switch` statement

Listing

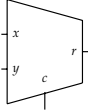
```
1 switch( c ) {  
2   case 0 : r0 = x; break;  
3   case 1 : r1 = x; break;  
4   case 2 : r2 = x; break;  
5   case 3 : r3 = x; break;  
6 }
```

acts similarly to a 4-output demultiplexer.

Notes:

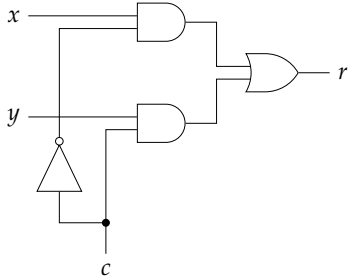
Part 2: special-purpose building blocks (3)

Choice

Definition			
The behaviour of a 2-input, 1-bit multiplexer component			
			
is described by the truth table			
c	x	y	r
0	0	?	0
0	1	?	1
1	?	0	0
1	?	1	1

which can be used to derive the following implementation:

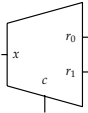
$$r = (\neg c \wedge x) \vee (c \wedge y)$$

Circuit			
			

Notes:

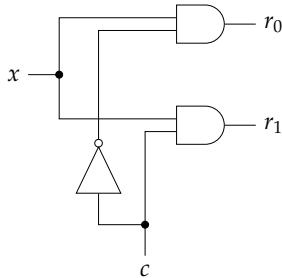
Part 2: special-purpose building blocks (4)

Choice

Definition			
The behaviour of a 2-output, 1-bit demultiplexer component			
			
is described by the truth table			
c	x	r_1	r_0
0	0	?	0
0	1	?	1
1	0	0	?
1	1	1	?

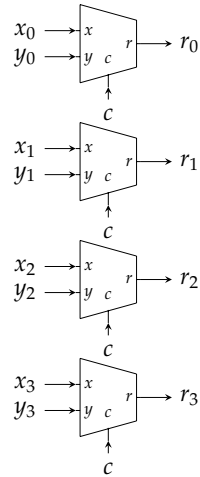
which can be used to derive the following implementation:

$$\begin{aligned} r_0 &= \neg c \wedge x \\ r_1 &= c \wedge x \end{aligned}$$

Circuit			
			

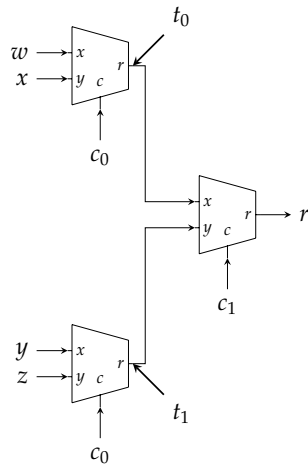
Notes:

Circuit (2-input, 4-bit multiplexer via independent replication)



Notes:

Circuit (4-input, 1-bit multiplexer via dependent replication)



Notes:

Part 2: special-purpose building blocks (7)

Addition

► Concept: addition.

1. a **half-adder**

- has 2 inputs: x and y ,
- computes the 2-bit result $x + y$,
- has 2 outputs: a sum s , and a carry-out co (which are the LSB and MSB of result),

while

2. a **full-adder**

- has 3 inputs: x and y plus a carry-in ci ,
- computes the 2-bit result $x + y + ci$,
- has 2 outputs: a sum s , and a carry-out co (which are the LSB and MSB of result),

where all inputs and outputs are 1-bit.

Notes:

Part 2: special-purpose building blocks (8)

Addition

Definition

The behaviour of a **half-adder** component



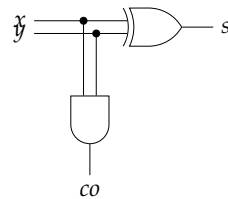
is described by the truth table

x	y	co	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

which can be used to derive the following implementation:

$$\begin{aligned} co &= x \wedge y \\ s &= x \oplus y \end{aligned}$$

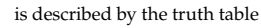
Circuit



Notes:

Addition

The behaviour of a **full-adder** component



which can be used to derive the following implementation:

Notes:

Addition

Notes:

Part 2: special-purpose building blocks (11)

Comparison

► Concept: comparison.

1. an equality comparator

- has 2 inputs x and y ,
- computes the 1 output as

$$r = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

while

2. a less-than comparator

- has 2 inputs x and y ,
- computes the 1 output as

$$r = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

where all inputs and outputs are 1-bit.

Notes:

Part 2: special-purpose building blocks (12)

Comparison

Definition

The behaviour of an **equality comparator** component



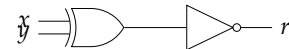
is described by the truth table

x	y	r
0	0	1
0	1	0
1	0	0
1	1	1

which can be used to derive the following implementation:

$$r = \neg(x \oplus y)$$

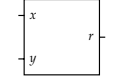
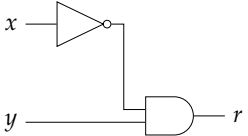
Circuit



Notes:

Part 2: special-purpose building blocks (13)

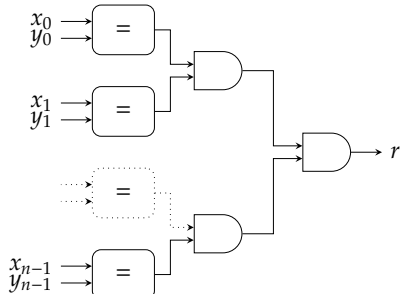
Comparison

Definition		Circuit															
The behaviour of a less-than comparator component  is described by the truth table <table border="1"><thead><tr><th>x</th><th>y</th><th>r</th></tr></thead><tbody><tr><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></tbody></table> which can be used to derive the following implementation: $r = \neg x \wedge y$		x	y	r	0	0	0	0	1	1	1	0	0	1	1	0	
x	y	r															
0	0	0															
0	1	1															
1	0	0															
1	1	0															

Notes:

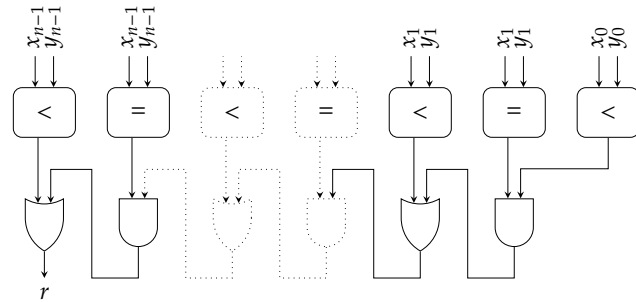
Part 2: special-purpose building blocks (14)

Comparison

Circuit (n -bit equality comparison)	
	

Notes:

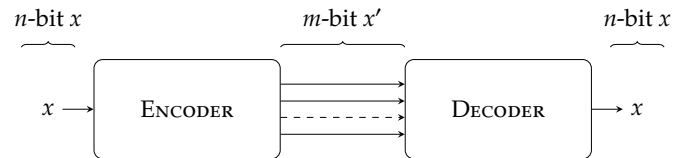
Circuit (n -bit less-than comparison)



Notes:

Part 2: special-purpose building blocks (16) Control

► **Concept:** translation, via **encoders** and **decoders**, i.e.,



or, more formally,

1. an n -to- m encoder translates an n -bit input into some m -bit code word, and
 2. an m -to- n decoder translates an m -bit code word back into the same n -bit output
- where if only one output (resp. input) is allowed to be 1 at a time, we call it a **one-of-many** encoder (resp. decoder).

Notes:

- **Example:**
1. to encode, take 4 inputs, say x_i for $0 \leq i < 4$, and produce a unsigned integer x' that determines which $x_i = 1$,
 2. to decode, take x' and set the correct $x_i = 1$ where for all $j \neq i$, $x'_j = 0$.

Notes:

Definition (example encoder)

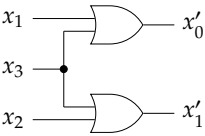
The example encoder is described by the truth table

x_3	x_2	x_1	x_0	x'_1	x'_0
0	0	0	1	0	0
0	0	1	0	0	1
0	1	0	0	1	0
1	0	0	0	1	1

which can be used to derive the following implementation:

$$\begin{aligned} x'_0 &= x_1 \vee x_3 \\ x'_1 &= x_2 \vee x_3 \end{aligned}$$

Circuit (example encoder)



Notes:

Definition (example decoder)

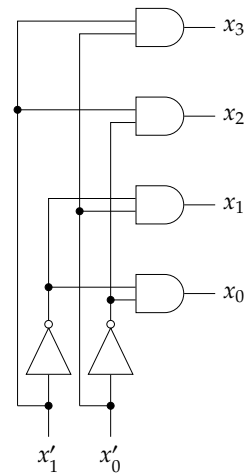
The example decoder is described by the truth table

x'_1	x'_0	x_3	x_2	x_1	x_0
0	0	0	0	0	1
0	1	0	0	1	0
1	0	0	1	0	0
1	1	1	0	0	0

which can be used to derive the following implementation:

$$\begin{aligned}
 x_0 &= \neg x'_0 \wedge \neg x'_1 \\
 x_1 &= x'_0 \wedge \neg x'_1 \\
 x_2 &= \neg x'_0 \wedge x'_1 \\
 x_3 &= x'_0 \wedge x'_1
 \end{aligned}$$

Circuit (example decoder)



Notes:

- **Problem:** if we break the rules and set both $x_1 = 1$ and $x_2 = 1$, the encoder fails by producing

$$\begin{aligned}
 x'_0 &= x_1 \vee x_3 = 1 \\
 x'_1 &= x_2 \vee x_3 = 1
 \end{aligned}$$

as the result.

- **Solution:** consider a **priority** encoder, where one input is given priority (or preference) over another.

Notes:

Example

Imagine we want to give x_j priority over each x_k for $j > k$, so x_2 over x_1 and x_0 for example:

x_3	x_2	x_1	x_0	x'_1	x'_0
0	0	0	1	0	0
0	0	1	?	0	1
0	1	?	?	1	0
1	?	?	?	1	1

Now, although potentially $x_0 = 1$ or $x_1 = 1$ the output gives priority to x_2 : as long as $x_2 = 1$ and $x_3 = 0$, the output will be $x'_0 = 0$ and $x'_1 = 1$ irrespective of x_0 and x_1 .

Notes:

Part 3: general-purpose derivation (1)
Method #1

Algorithm

Input: A truth table for some Boolean function f , with n inputs and 1 output

Output: A Boolean expression e that implements f

First let I_j be a synonym for the j -th input, for $0 \leq j < n$, and O be a synonym for the output.

- Find a set T such that $i \in T$ iff. $O = 1$ in the i -th row of the truth table.
- For each $i \in T$, form a term t_i by AND'ing together all the inputs while following two rules:

- 2.1 if $I_j = 1$ in the i -th row, then we use

$$I_j$$

as is, but

- 2.2 if $I_j = 0$ in the i -th row, then we use

$$\neg I_j.$$

3. An expression implementing the function is then formed by OR'ing together all the terms, i.e.,

$$e = \bigvee_{i \in T} t_i,$$

which is in SoP form.

Notes:

Example

Consider the example of deriving an expression for XOR, i.e.,

$$r = f(x, y) = x \oplus y,$$

a function described by the following truth table:

x	y	r
0	0	0
0	1	1
1	0	1
1	1	0

Notes:

Example

Consider the example of deriving an expression for XOR, i.e.,

$$r = f(x, y) = x \oplus y,$$

a function described by the following truth table:

x	y	r
0	0	0
0	1	1
1	0	1
1	1	0

Based on the fact there are $n = 2$ inputs and 1 output, we can use

$$\begin{array}{lcl} I_0 & \equiv & x \\ I_1 & \equiv & y \\ O & \equiv & r \end{array}$$

Notes:

Example

Consider the example of deriving an expression for XOR, i.e.,

$$r = f(x, y) = x \oplus y,$$

a function described by the following truth table:

x	y	r	
0	0	0	
0	1	1	$\rightsquigarrow i = 1$
1	0	1	$\rightsquigarrow i = 2$
1	1	0	

The algorithm steps are as follows:

1. Clearly $T = \{1, 2\}$, because

- ▶ $0 \equiv r = 1$ in rows 1 and 2,
- ▶ $0 \equiv r = 0$ in rows 0 and 3.

Example

Consider the example of deriving an expression for XOR, i.e.,

$$r = f(x, y) = x \oplus y,$$

a function described by the following truth table:

x	y	r	
0	0	0	
0	1	1	$\rightsquigarrow t_1 = \neg x \wedge y$
1	0	1	$\rightsquigarrow t_2 = x \wedge \neg y$
1	1	0	

The algorithm steps are as follows:

2. Each term t_i for $i \in T = \{1, 2\}$ is formed as follows:

- ▶ For $i = 1$, we find
 - ▶ $I_0 \equiv x = 0$ and so we use $\neg x$,
 - ▶ $I_1 \equiv y = 1$ and so we use yand hence form the term $t_1 = \neg x \wedge y$.
- ▶ For $i = 2$, we find
 - ▶ $I_0 \equiv x = 1$ and so we use x ,
 - ▶ $I_1 \equiv y = 0$ and so we use $\neg y$and hence form the term $t_2 = x \wedge \neg y$.

Notes:

Notes:

Part 3: general-purpose derivation (2)

Method #1

Example

Consider the example of deriving an expression for XOR, i.e.,

$$r = f(x, y) = x \oplus y,$$

a function described by the following truth table:

x	y	r
0	0	0
0	1	1
1	0	1
1	1	0

$\rightsquigarrow t_1 = \neg x \wedge y$
 $\rightsquigarrow t_2 = x \wedge \neg y$

The algorithm steps are as follows:

- The expression implementing the function is therefore

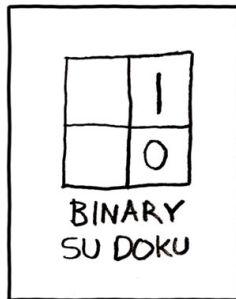
$$\begin{aligned} e &= \bigvee_{i \in T} t_i \\ &= \bigvee_{i \in \{1,2\}} t_i \\ &= (\neg x \wedge y) \vee (x \wedge \neg y) \end{aligned}$$

which is in SoP form.

Notes:

Part 3: general-purpose derivation (3)

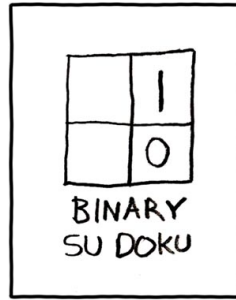
Method #2: Karnaugh map



Notes:

Part 3: general-purpose derivation (3)

Method #2: Karnaugh map



► Idea:

$$\begin{aligned}(x \wedge y) \vee (x \wedge \neg y) &\equiv x \wedge (y \vee \neg y) && \text{(distribution)} \\ &\equiv x \wedge 1 && \text{(inverse)} \\ &\equiv x && \text{(identity)}\end{aligned}$$

<https://xkcd.com/74>

Part 3: general-purpose derivation (4)

Method #2: Karnaugh map

Algorithm

Input: A truth table for some Boolean function f , with n inputs and 1 output

Output: A Boolean expression e that implements f

1. Draw a rectangular $(p \times q)$ -element grid, st.
 - 1.1 $p \equiv q \equiv 0 \pmod{2}$, and
 - 1.2 $p \cdot q = 2^n$and each row and column represents one input combination; order rows and columns according to a **Gray code**.
2. Fill the grid elements with the output corresponding to inputs for that row and column.
3. Cover rectangular groups of adjacent 1 elements which are of total size 2^m for some m , aiming for
 - 3.1 *bigger* groups, and
 - 3.2 *fewer* groupsbecause this produces a simpler expression; groups can “wrap around” edges of the grid, and/or overlap.
4. Translate each group into one term of an SoP form Boolean expression e .

Notes:

Notes:

Example					
Natural sequence			Gray code sequence		
$\langle 0, 0, 0, 0 \rangle$	$\mapsto$	$0_{(10)}$	$\langle 0, 0, 0, 0 \rangle$	$\mapsto$	$0_{(10)}$
$\langle 1, 0, 0, 0 \rangle$	$\mapsto$	$1_{(10)}$	$\langle \textcolor{red}{1}, 0, 0, 0 \rangle$	$\mapsto$	$1_{(10)}$
$\langle 0, 1, 0, 0 \rangle$	$\mapsto$	$2_{(10)}$	$\langle 1, \textcolor{red}{1}, 0, 0 \rangle$	$\mapsto$	$3_{(10)}$
$\langle 1, 1, 0, 0 \rangle$	$\mapsto$	$3_{(10)}$	$\langle \textcolor{red}{0}, 1, 0, 0 \rangle$	$\mapsto$	$2_{(10)}$
$\langle 0, 0, 1, 0 \rangle$	$\mapsto$	$4_{(10)}$	$\langle 0, 1, \textcolor{red}{1}, 0 \rangle$	$\mapsto$	$6_{(10)}$
$\langle 1, 0, 1, 0 \rangle$	$\mapsto$	$5_{(10)}$	$\langle 0, \textcolor{red}{0}, 1, 0 \rangle$	$\mapsto$	$4_{(10)}$
$\langle 0, 1, 1, 0 \rangle$	$\mapsto$	$6_{(10)}$	$\langle \textcolor{red}{1}, 0, 1, 0 \rangle$	$\mapsto$	$5_{(10)}$
$\langle 1, 1, 1, 0 \rangle$	$\mapsto$	$7_{(10)}$	$\langle 1, \textcolor{red}{1}, 1, 0 \rangle$	$\mapsto$	$7_{(10)}$
$\vdots$			$\vdots$		

Notes:

Example					
Consider an example 4-input, 1-output function:					
w	x	y	z	r	
0	0	0	0	0	1
0	0	0	1	0	1
0	0	1	0	0	1
0	0	1	1	0	0
0	1	0	0	0	1
0	1	0	1	0	1
0	1	1	0	0	0
0	1	1	1	0	0
1	0	0	0	1	1
1	0	0	1	1	0
1	0	1	0	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	0	1	0	0
1	1	1	0	0	0
1	1	1	1	1	1

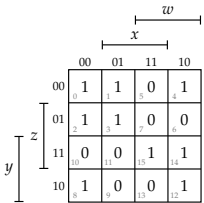
Notes:

- The first two steps are simple: drawing a grid of an appropriate size then filling it with entries from the truth table are both trivial, even though they still demand some care.
- Forming the groups is a little harder, in the sense that their “quality” (i.e., number and size) dictates how optimised the resulting expression will be. This example has some non-intuitive cases: the blue group might not be obvious for example, *but* is within the rules (although it is sort of inside-out, it is rectangular and a power-of-two in size).
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 - The red group spans columns 0 and 1 and rows 0 and 1; provided $w = 0$ and $y = 0$ we specify *just* those cells, so the expression is $\neg w \wedge \neg y$. That is, $w = 0$ restricts us to columns 0 and 1 (columns 2 and 3 have $w = 1$) and $y = 0$ restricts us to rows 0 and 1 (rows 2 and 3 have $y = 1$). Note that the values of x and z don't matter, i.e., cells in the group hold the value 1 regardless of x and z .
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Example

Consider an example 4-input, 1-output function:

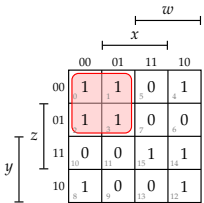
w	x	y	z	r
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1



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0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1



Each group translates into one term of the SoP form expression

$$r = (\neg w \wedge \neg y)$$

Notes:

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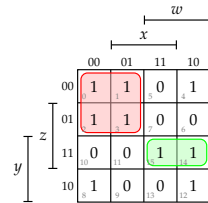
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0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1



Each group translates into one term of the SoP form expression

$$r = (\neg w \wedge \neg y \wedge z) \vee (w \wedge y \wedge z)$$

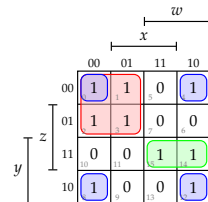
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0	1	0	0	1
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0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1



Each group translates into one term of the SoP form expression

$$r = (\neg w \wedge \neg y \wedge z) \vee (w \wedge y \wedge z) \vee (\neg x \wedge \neg z)$$

Notes:

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Example

Consider an example 3-input, 1-output function:

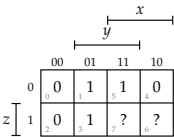
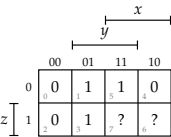
<i>x</i>	<i>y</i>	<i>z</i>	<i>r</i>
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	?
1	1	0	1
1	1	1	?

- Notes:
- Note the differing shape of this grid versus the previous example: since there are $n = 3$ variables, we set $p = 2$ and $q = 4$ such that the grid contains $2 \cdot 4 = 2^3 = 8$ cells.
 - Adopting the same approach as the previous example, by not ignoring the don't care entries (i.e., assuming they are 0) we have two groups. However, opting to treat one of them as a 1 (which is fine: by definition we don't care what the output is) we only have one:
 - The red group spans column 1 and rows 0 and 1; provided $x = 0$ and $y = 1$ we specify *just* those cells, so the expression is $\neg x \wedge y$. That is, $x = 0$ and $y = 1$ restricts us to column 1 (columns 0, 2 and 3 have at least one of $x = 1$ or $y = 0$) which is all we need because the group spans *all* rows. Note that the value of z doesn't matter, i.e., cells in the group hold the value 1 regardless of z .
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<i>x</i>	<i>y</i>	<i>z</i>	<i>r</i>
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	?
1	1	0	1
1	1	1	?

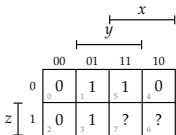
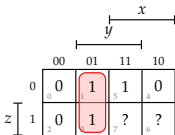


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1	1	1	?



Each group translates into one term of the SoP form expressions

$$r = (\neg x \wedge y)$$

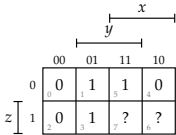
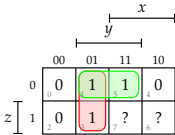
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0	0	1	0
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Each group translates into one term of the SoP form expressions

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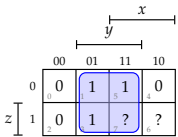
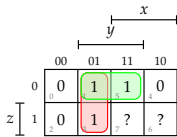
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Example

Consider an example 3-input, 1-output function:

<i>x</i>	<i>y</i>	<i>z</i>	<i>r</i>
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	?
1	1	0	1
1	1	1	?



Each group translates into one term of the SoP form expressions

$r = (\neg x \wedge y) \vee (y \wedge \neg z)$ $r = y$

where effective use of don't care states yields a clear improvement!

Notes:

- Note the differing shape of this grid versus the previous example: since there are $n = 3$ variables, we set $p = 2$ and $q = 4$ such that the grid contains $2 \cdot 4 = 2^3 = 8$ cells.
- Adopting the same approach as the previous example, by not ignoring the don't care entries (i.e., assuming they are 0) we have two groups. However, opting to treat one of them as a 1 (which is fine: by definition we don't care what the output is) we only have one:
 - The red group spans column 1 and rows 0 and 1; provided $x = 0$ and $y = 1$ we specify *just* those cells, so the expression is $\neg x \wedge y$. That is, $x = 0$ and $y = 1$ restricts us to column 1 (columns 0, 2 and 3 have at least one of $x = 1$ or $y = 0$) which is all we need because the group spans *all* rows. Note that the value of z doesn't matter, i.e., cells in the group hold the value 1 regardless of z .
 - The green group spans columns 1 and 2, in row 0; provided $y = 1$ and $z = 0$ we specify *just* those cells, so the expression is $y \wedge \neg z$. That is, $y = 1$ restricts us to columns 1 and 2 (columns 0 and 3 have $y = 0$) and $z = 0$ restricts us to row 0 (row 1 has $z = 1$). Note that the value of x doesn't matter, i.e., cells in the group hold the value 1 regardless of x .
 - The blue group spans columns 1 and 2 and rows 0 and 1; provided $y = 1$ we specify *just* those cells, so the expression is y . That is, $y = 1$ restricts us to columns 1 and 2 (columns 0 and 3 have $y = 0$) which is all we need because the group spans *all* rows. Note that the values of x and z don't matter, i.e., cells in the group hold the value 1 regardless of x and z .

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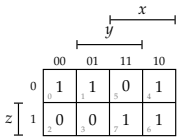
Notes:

- The red group spans columns 2 and 3, in row 1; provided $x = 1$ and $z = 1$ we specify *just* those cells, so the expression is $x \wedge z$. $x = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $x = 0$) and $z = 1$ restricts us to row 1 (row 0 has $z = 0$). Note that the value of y doesn't matter, i.e., cells in the group hold the value 1 regardless of y .
 - The green group spans column 3 and rows 0 and 1; provided $x = 1$ and $y = 0$ we specify *just* those cells, so the expression is $x \wedge \neg y$. That is, $x = 1$ and $y = 0$ restricts us to column 3 (columns 0, 1 and 2 have at least one of $x = 0$ or $y = 1$) which is all we need because the group spans *all* rows. Note that the value of z doesn't matter, i.e., cells in the group hold the value 1 regardless of z .
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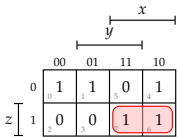
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Each group translates into one term of the SoP form expression

$$r = (\quad x \quad \quad \wedge \quad z \quad)$$

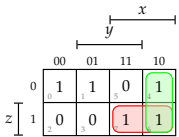
Notes:

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0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1



Each group translates into one term of the SoP form expression

$$r = (\begin{matrix} x \\ x \end{matrix} \wedge \begin{matrix} z \\ \neg y \end{matrix}) \vee$$

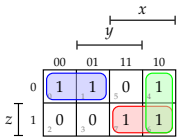
Notes:

- The red group spans columns 2 and 3, in row 1; provided $x = 1$ and $z = 1$ we specify *just* those cells, so the expression is $x \wedge z$. $x = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $x = 0$) and $z = 1$ restricts us to row 1 (row 0 has $z = 0$). Note that the value of y doesn't matter, i.e., cells in the group hold the value 1 regardless of y .
- The green group spans column 3 and rows 0 and 1; provided $x = 1$ and $y = 0$ we specify *just* those cells, so the expression is $x \wedge \neg y$. That is, $x = 1$ and $y = 0$ restricts us to column 3 (columns 0, 1 and 2 have at least one of $x = 0$ or $y = 1$) which is all we need because the group spans *all* rows. Note that the value of z doesn't matter, i.e., cells in the group hold the value 1 regardless of z .
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0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
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Each group translates into one term of the SoP form expression

$$r = (\begin{matrix} x \\ x \\ \neg x \end{matrix} \wedge \begin{matrix} z \\ \neg y \\ \neg z \end{matrix}) \vee$$

Notes:

- The red group spans columns 2 and 3, in row 1; provided $x = 1$ and $z = 1$ we specify *just* those cells, so the expression is $x \wedge z$. $x = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $x = 0$) and $z = 1$ restricts us to row 1 (row 0 has $z = 0$). Note that the value of y doesn't matter, i.e., cells in the group hold the value 1 regardless of y .
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Example

Consider an example 4-input, 2-output function:

w	x	y	z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	?	?
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	?	?

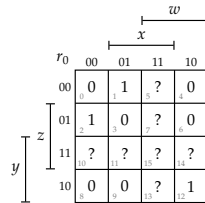
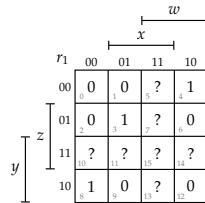
Notes:

- The red group spans columns 2 and 3 in row 0; provided $w = 1, y = 0$ and $z = 0$ we specify *just* those cells, so the expression is $w \wedge \neg y \wedge \neg z$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 0$ and $z = 0$ restricts us to row 0 (rows 1, 2 and 3 have at least one of $y = 1$ or $z = 1$). Note that the value of x doesn't matter, i.e., cells in the group hold the value 1 regardless of x .
 - The green group spans column 1 and rows 2 and 3; provided $w = 0, x = 0$ and $y = 1$ we specify *just* those cells, so the expression is $\neg w \wedge \neg x \wedge y$. That is, $w = 0$ and $x = 0$ restricts us to column 0 (columns 1, 2 and 3 have at least one of $w = 1$ or $x = 1$) and $y = 1$ restricts us to rows 2 and 3 (rows 0 and 1 have $y = 0$). Note that the value of z doesn't matter, i.e., cells in the group hold the value 1 regardless of z .
 - The blue group spans columns 1 and 2 and rows 1 and 2; provided $x = 1$ and $z = 1$ we specify *just* those cells, so the expression is $x \wedge z$. That is, $x = 1$ restricts us to columns 1 and 2 (columns 0 and 3 have $x = 0$) and $z = 1$ restricts us to rows 1 and 2 (rows 0 and 3 have $z = 0$). Note that the value of w and y don't matter, i.e., cells in the group hold the value 1 regardless of w and y .
 - The magenta group spans columns 1 and 2 in row 0; provided $x = 1, y = 0$ and $z = 0$ we specify *just* those cells, so the expression is $x \wedge \neg y \wedge \neg z$. That is, $x = 1$ restricts us to columns 1 and 2 (columns 0 and 3 have $x = 0$) and $y = 0$ and $z = 0$ restricts us to row 0 (rows 1, 2 and 3 have at least one of $y = 1$ or $z = 1$). Note that the value of w doesn't matter, i.e., cells in the group hold the value 1 regardless of w .
 - The yellow group spans column 0 and rows 1 and 2; provided $w = 0, x = 0$ and $z = 1$ we specify *just* those cells, so the expression is $\neg w \wedge \neg x \wedge z$. That is, $w = 0$ and $x = 0$ restricts us to column 0 (columns 1, 2 and 3 have at least one of $w = 1$ or $x = 1$) and $y = 1$ restricts us to rows 1 and 2 (rows 0 and 1 have $y = 0$). Note that the value of w doesn't matter, i.e., cells in the group hold the value 1 regardless of w .
 - The orange group spans columns 2 and 3 and rows 2 and 3; provided $w = 1$ and $y = 1$ we specify *just* those cells, so the expression is $w \wedge y$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 1$ restricts us to rows 2 and 3 (rows 0 and 1 have $y = 0$). Note that the value of x and z don't matter, i.e., cells in the group hold the value 1 regardless of x and z .

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0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	?	?
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	?	?



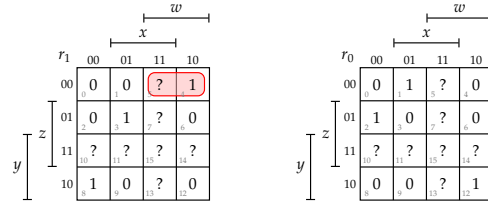
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 - The orange group spans columns 2 and 3 and rows 2 and 3; provided $w = 1$ and $y = 1$ we specify *just* those cells, so the expression is $w \wedge y$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 1$ restricts us to rows 2 and 3 (rows 0 and 1 have $y = 0$). Note that the value of x and z don't matter, i.e., cells in the group hold the value 1 regardless of x and z .

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0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	?	?
0	1	0	0	?	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	?	?



Each group translates into one term of the SoP form expressions

$$r_1 = (\quad w \quad \wedge \quad \neg y \quad \wedge \quad \neg z \quad)$$

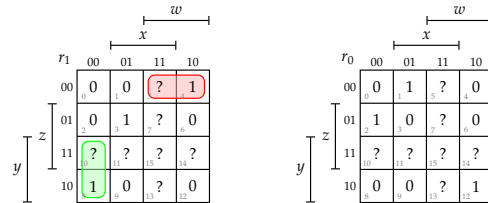
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0	0	1	1	?	?
0	1	0	0	?	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	?	?



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$$r_1 = (\quad w \quad \wedge \quad \neg y \quad \wedge \quad \neg z \quad) \vee (\quad y \quad \wedge \quad \neg w \quad \wedge \quad \neg x \quad)$$

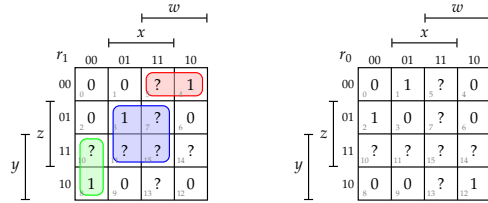
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- The orange group spans columns 2 and 3 and rows 2 and 3; provided $w = 1$ and $y = 1$ we specify *just* those cells, so the expression is $w \wedge y$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 1$ restricts us to rows 2 and 3 (rows 0 and 1 have $y = 0$). Note that the value of x and z don't matter, i.e., cells in the group hold the value 1 regardless of x and z .

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Consider an example 4-input, 2-output function:

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0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	?	?
0	1	0	0	?	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	?	?



Each group translates into one term of the SoP form expressions

$$r_1 = (w \wedge \neg y \wedge \neg z) \vee (y \wedge \neg w \wedge \neg x) \vee (x \wedge z)$$

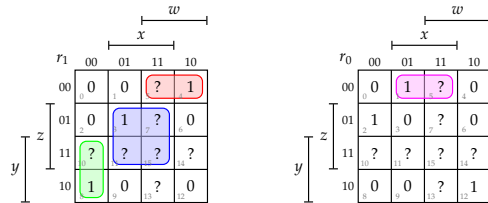
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0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
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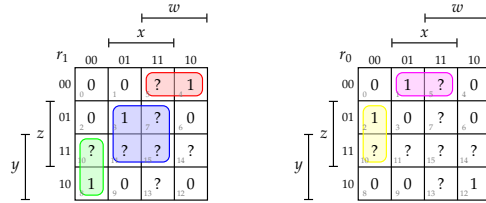
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$$r_1 = (\begin{matrix} w & \wedge & \neg y & \wedge & \neg z \\ y & \wedge & \neg w & \wedge & \neg x \\ x & \wedge & z \end{matrix}) \vee$$

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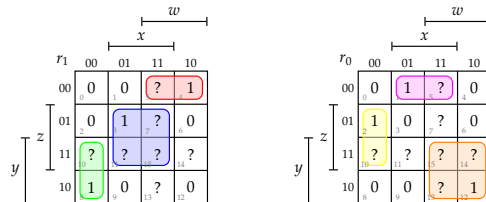
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Conclusions

► Take away points:

1. **Bad news:** even (relatively) simple problems present numerous challenges, e.g.,
 - how do we describe what the design should do?
 - how do we structure the design?
 - what sort of standard cell library do we use?
 - do we aim for the fewest gates?
 - do we aim for shortest critical path?
 - how do we cope with propagation delay and fan-out?
 - ...
2. **Good news:** the approaches we've covered, i.e.,
 - special-purpose design patterns,
 - special-purpose building-block components,
 - general-purpose derivation and optimisation of Boolean expressions,allows us to address such challenges: in combination, they support development of effective (combinatorial) design and implementation.
3. **Good news:** in many cases, use of appropriate **Electronic Design Automation (EDA)** tools can provide (semi-)automatic solutions.

Notes:

Additional Reading

- *Wikipedia: Combinational logic*. URL: https://en.wikipedia.org/wiki/Combinational_logic.
- D. Page. “Chapter 2: Basics of digital logic”. In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009.
- W. Stallings. “Chapter 11: Digital logic”. In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013.
- A.S. Tanenbaum and T. Austin. “Section 3.2.2: Combinatorial circuits”. In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012.

Notes:

References

- [1] [Wikipedia: Combinational logic](https://en.wikipedia.org/wiki/Combinational_logic). URL: https://en.wikipedia.org/wiki/Combinational_logic (see p. 135).
- [2] D. Page. “Chapter 2: Basics of digital logic”. In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009 (see p. 135).
- [3] W. Stallings. “Chapter 11: Digital logic”. In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013 (see p. 135).
- [4] A.S. Tanenbaum and T. Austin. “Section 3.2.2: Combinatorial circuits”. In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012 (see p. 135).

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