

- **Agenda:** introduce the topic [4, Part 1] of

finite automata $\equiv$ **Finite State Machines (FSMs)**

via

1. an “in theory”, i.e., concept-oriented perspective, and
2. an “in practice”, i.e., perspective, spanning
 - 2.1 general application, and
 - 2.2 specific implementation in sequential logic design.

Part 1: in theory (1)

Definition

An **alphabet** is a non-empty set of **symbols**.

Definition

A **string** X with respect to some alphabet Σ is a sequence, of finite length, whose elements are members of Σ , i.e.,

$$X = \langle X_0, X_1, \dots, X_{n-1} \rangle$$

for some n such that $X_i \in \Sigma$ for $0 \leq i < n$; if n is zero, we term X the **empty string** and denote it ϵ . It can be useful, and is common to write elements in human-readable form termed a **string literal**: this basically means writing them from right-to-left without any associated notation (e.g., brackets or commas).

Definition

A **language** Λ is a set of strings.

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 - ▶ Once the input is exhausted, C halts: depending on the state it halts in, we say either
 1. C accepts (or recognises) the input string
 2. C rejects the input string
 - ▶ For a language Λ of all possible input strings C could accept, we say
$$C \text{ accepts (or recognises) } \Lambda \equiv \Lambda \text{ is the language of } C$$
and use Λ to classify C ...

Part 1: in theory (4)

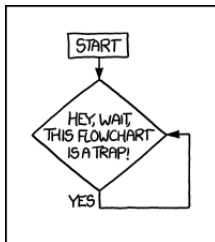
Definition

less powerful

more powerful

Machine	Combinatorial logic	Finite automaton	Push-down automaton	Linear-bounded automaton	Turing machine
Memory		0 stacks	1 stacks	2 stacks	2 stacks
Language		regular	context free	context sensitive	recursively enumerable
Grammar		regular ($X \rightarrow x$ or $X \rightarrow xY$)	context free ($X \rightarrow \gamma$)	context sensitive ($\alpha X \beta \rightarrow \alpha \gamma \beta$)	unrestricted ($\alpha \rightarrow \beta$)
Chomsky-Schützenberger hierarchy		type-3	type-2	type-1	type-0

Part 1: in theory (5)



Definition

A (deterministic) **Finite State Machine (FSM)** is a tuple

$$C = (S, s, A, \Sigma, \Gamma, \delta, \omega)$$

including

1. S , a finite set of **states** that includes a **start state** $s \in S$,
2. $A \subseteq S$, a finite set of **accepting states**,
3. an **input alphabet** Σ and an **output alphabet** Γ ,
4. a **transition function**

$$\delta : S \times \Sigma \rightarrow S$$

and

5. an **output function**

$$\omega : S \rightarrow \Gamma$$

in the case of a **Moore FSM**, or

$$\omega : S \times \Sigma \rightarrow \Gamma$$

in the case of a **Mealy FSM**,

noting an **empty** input denoted ϵ allows a transition that can *always* occur.

- **Problem:** design an FSM that decides whether a binary sequence X has an odd number of 1 elements in it.

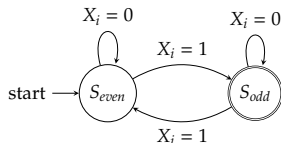
Part 1: in theory (7)

► Solution:

Algorithm (tabular)

Q	δ	
	Q' $X_i = 0$	$X_i = 1$
S_{even}	S_{even}	S_{odd}
S_{odd}	S_{odd}	S_{even}

Algorithm (diagram)



where, e.g.,

- for the input string $X = \langle 1, 0, 1, 1 \rangle$ the transitions are

$$\rightsquigarrow S_{even} \xrightarrow{X_0=1} S_{odd} \xrightarrow{X_1=0} S_{odd} \xrightarrow{X_2=1} S_{even} \xrightarrow{X_3=1} S_{odd}$$

so the input is accepted (i.e., has an odd number of 1 elements).

- for the input string $X = \langle 0, 1, 1, 0 \rangle$ the transitions are

$$\rightsquigarrow S_{even} \xrightarrow{X_0=0} S_{even} \xrightarrow{X_1=1} S_{odd} \xrightarrow{X_2=1} S_{even} \xrightarrow{X_3=0} S_{even}$$

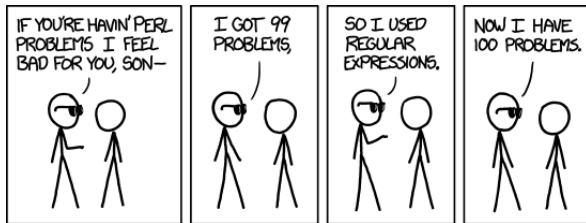
so the input is rejected (i.e., has an even number of 1 elements).

Part 2.1: in practice, application (1)

Example #1: regular expressions + grep $\leadsto$ FSMs as recognisers

► Context:

► -ve perspective:



► +ve perspective: we could say that

arithmetic expression

evaluate



number

regular expression

evaluate



language

so a regular expression (or regex) can be used as

1. a pattern used to describe or generate a language, *or*
2. a pattern used to identify (i.e., match) members of a language.

Part 2.1: in practice, application (3)

Example #1: regular expressions + `grep` $\leadsto$ FSMs as recognisers

► **Example** [4, Example 1.53]: if $\Sigma = \{0, 1\}$, then

$$0^*10^* \equiv \left\{ s \mid \begin{array}{l} s \text{ is a string containing} \\ \text{a single } 1 \end{array} \right\}$$

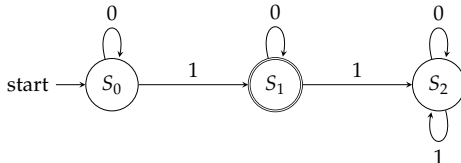
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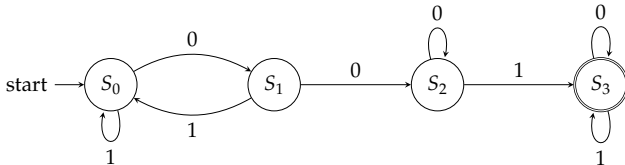
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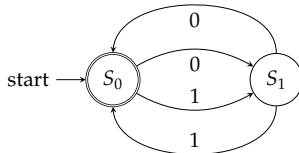
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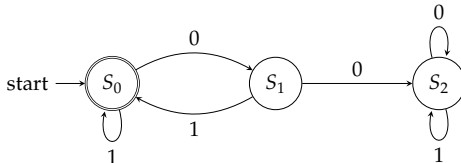
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► **Example** [4, Example 1.53]: if $\Sigma = \{'a', 'b', \dots, 'z'\}$, then

$$\text{grep -E '.*to+.*'} \equiv \left\{ s \mid \begin{array}{l} s \text{ is a line read from stdin containing} \\ \text{a 't' followed by at least one 'o' character} \end{array} \right\}$$

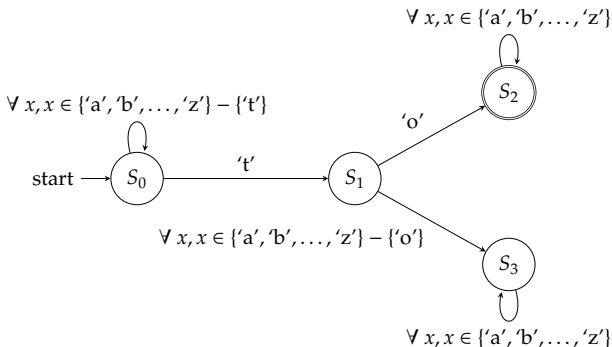
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```
1 forall lines X read from stdin do
2   | Q ← s
3   | for i = 0 upto n - 1 do
4     | Q ← δ(Q, Xi)
5   | end
6   | if Q ∈ A then
7     | print line X to stdout
8   | end
9 end
```

Part 2.1: in practice, application (3)

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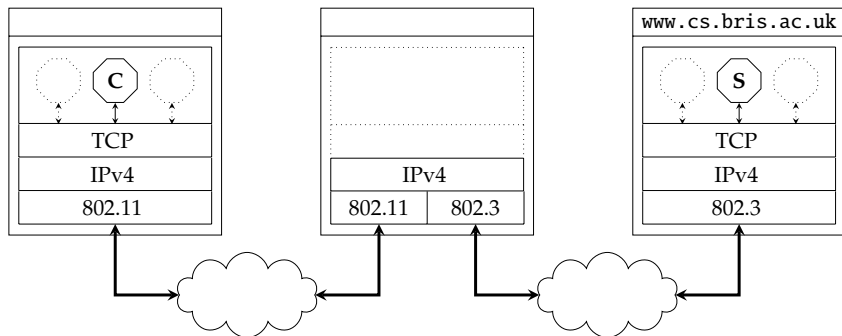
which can be (sort of, with a significant caveat) realised using

```
1 void grep() {
2   char X[ 1024 ];
3
4   while( NULL != fgets( X, 1024, stdin ) ) {
5     int n = strlen( X ), Q = start;
6
7     if( X[ n - 1 ] == '\n' ) {
8       X[ n - 1 ] = '\0'; n--;
9     }
10
11    for( int i = 0; i < n; i++ ) {
12      Q = delta[ Q ][ X[ i ] ];
13    }
14
15    if( accept[ Q ] ) {
16      fprintf( stdout, "%s\n", X );
17    }
18  }
19 }
```

Part 2.1: in practice, application (4)

Example #2: networked communication via TCP $\leadsto$ FSMs as controllers

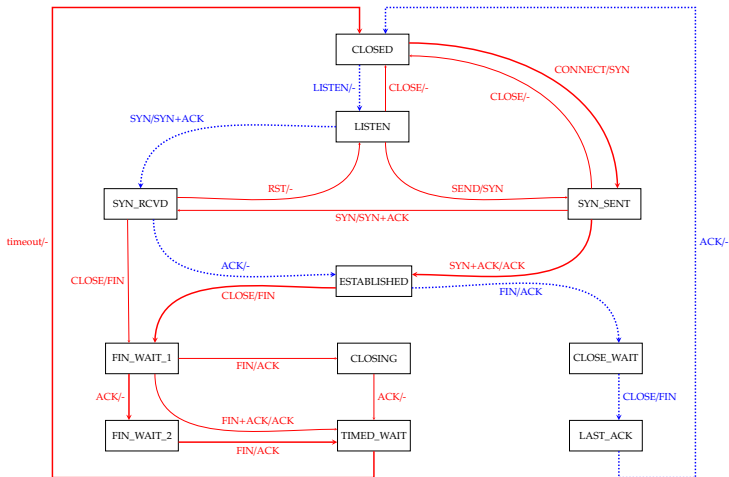
► Context:



Part 2.1: in practice, application (5)

Example #2: networked communication via TCP $\leadsto$ FSMs as controllers

► Example:



Part 2.1: in practice, application (6)

Example #3: typical video game “loop” $\leadsto$ FSMs as systems

► Context:



Algorithm

```
1 reset the game state
2 while  $\neg$  game over do
3   | read control pad (e.g., check if button pressed)
4   | update game state (e.g., move player)
5   | produce graphics and/or sound
6 end
```

Part 2.1: in practice, application (6)

Example #3: typical video game “loop” $\leadsto$ FSMs as systems

► Context:



Algorithm

```
1  $Q \leftarrow s$ 
2 while  $Q \notin A$  do
3    $X_i \leftarrow$  control pad
4    $Q \leftarrow \delta(Q, X_i)$ 
5    $\{\text{graphics, sound}\} \leftarrow \omega(Q)$ 
6 end
```

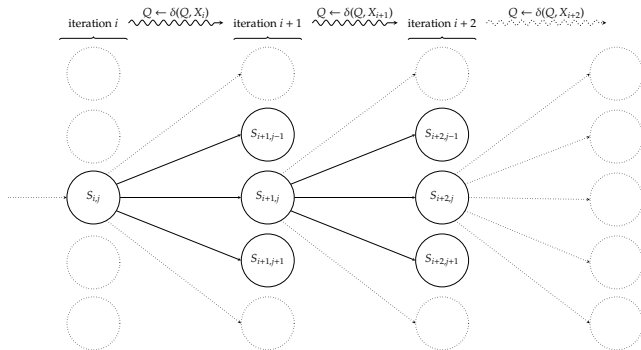
Part 2.1: in practice, application (7)

Example #3: typical video game "loop" $\leadsto$ FSMs as systems

► Example:

iterations of game loop $\leadsto$ game tree
 $\simeq$ state space

i.e.,



which is most obvious with respect to turn-based games (e.g., chess).

► Recall:

Definition

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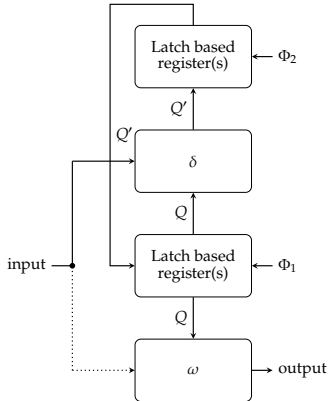
noting an **empty** input denoted ϵ allows a transition that can *always* occur.

Part 2.2: in practice, implementation (2)

Design framework

► Concept:

Algorithm (latch version)



► Note that

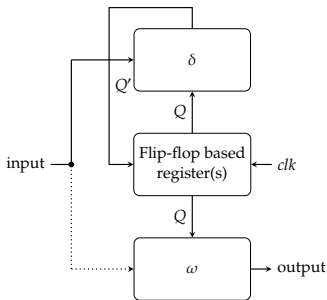
1. the state is retained in a register (i.e., a group of latches, resp. flip-flops),
 2. δ and ω are simply combinatorial logic,
 3. within the current clock cycle
 - ω computes the output from the current state and input, and
 - δ computes the next state from the current state and input,
 4. the next state is latched by an appropriate feature (i.e., level, resp. edge) in the clock
- i.e., it's a *computer* we can *build*!

Part 2.2: in practice, implementation (3)

Design framework

► Concept:

Algorithm (flip-flop version)



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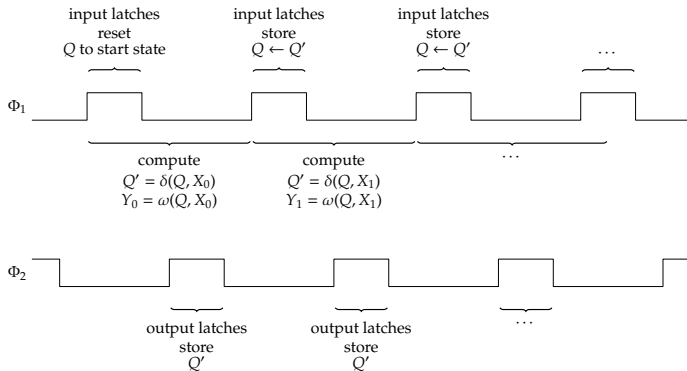
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Part 2.2: in practice, implementation (4)

Design framework

► Concept:

Example (latch version: input $X = \langle X_0, X_1, \dots \rangle$, output $Y = \langle Y_0, Y_1, \dots \rangle$)

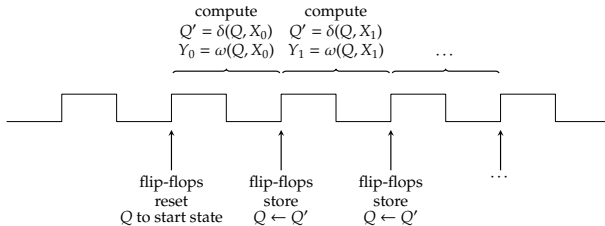


Part 2.2: in practice, implementation (5)

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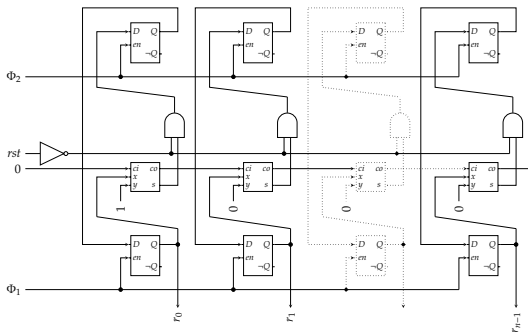


Part 2.2: in practice, implementation (6)

Design framework

- **Concept:** this *should* sound familiar, because from

Circuit (latch version)



it *now* becomes clear that

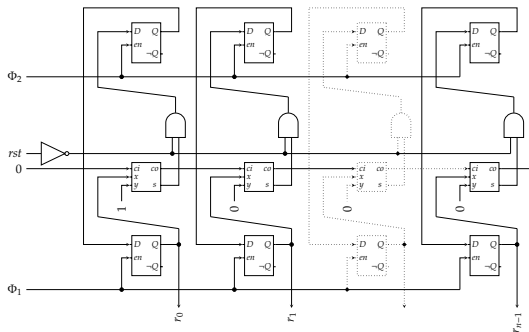
- 2^n states, labelled S_0 through S_{2^n-1} ; state S_i represented as (unsigned) n -bit integer i ,
- the start state is $s = S_0$ and there are no accepting states (so $A = \emptyset$),

Part 2.2: in practice, implementation (6)

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- the δ function is

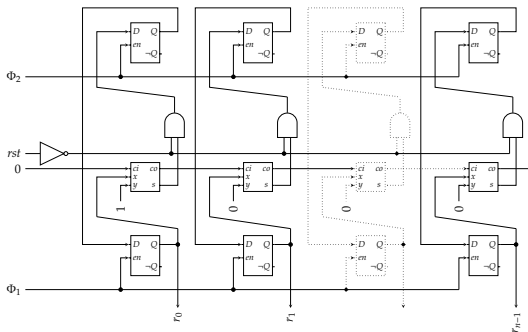
$$Q' \leftarrow \delta(Q, rst) = \begin{cases} Q + 1 & (\text{mod } 2^n) & \text{if } rst = 0 \\ 0 & & \text{if } rst = 1 \end{cases}$$

Part 2.2: in practice, implementation (6)

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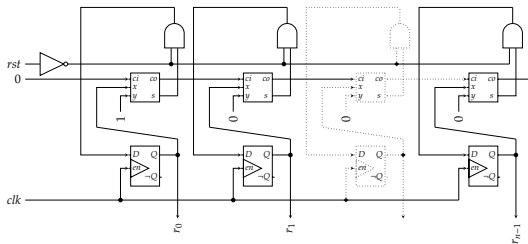
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Part 2.2: in practice, implementation (7)

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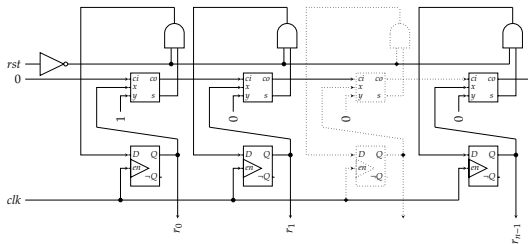
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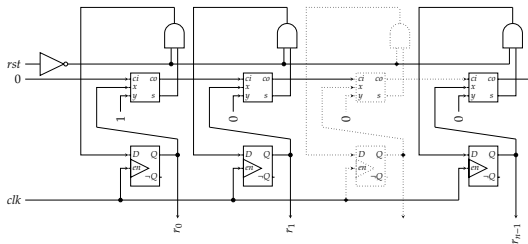
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it *now* becomes clear that

- the ω function is $r \leftarrow \omega(Q) = Q$.

- **Concept:** to solve a concrete problem, we follow a (fairly) standard sequence of steps

Algorithm

1. Count the number of states required, and give each state an abstract label.
2. Describe the state transition and output functions using a tabular or diagrammatic approach.
3. Perform state assignment, i.e., decide how concrete values will represent the abstract labels, allocating appropriate register(s) to hold the state.
4. Express the functions δ and ω as (optimised) Boolean expressions, i.e., combinatorial logic.
5. Place the registers and combinatorial logic into the framework.

noting that it's common to

- include a **reset** input that (re)initialises the FSM into the start state,
- replace the accepting state(s) with an **idle** or **error** state since “halting” doesn't make sense in hardware, and
- use the FSM to control an associated data-path using the outputs, rather than (necessarily) solve some problem outright.

Part 2.2: in practice, implementation (9)

Design process

► **Concept:** we can optimise the state representation based on use of it, e.g.,

1. a **binary encoding** represents the i -th of n states as a $(\lceil \log_2(n) \rceil)$ -bit unsigned integer i , e.g.,

$$\begin{aligned} S_0 &\mapsto \langle 0, 0, 0 \rangle \\ S_1 &\mapsto \langle 1, 0, 0 \rangle \\ S_2 &\mapsto \langle 0, 1, 0 \rangle \\ S_3 &\mapsto \langle 1, 1, 0 \rangle \\ S_4 &\mapsto \langle 0, 0, 1 \rangle \\ S_5 &\mapsto \langle 1, 0, 1 \rangle \end{aligned}$$

2. a **one-hot encoding** represents the i -th of n states as a sequence X such that $X_i = 1$ and $X_j = 0$ for $j \neq i$, e.g.,

$$\begin{aligned} S_0 &\mapsto \langle 1, 0, 0, 0, 0, 0 \rangle \\ S_1 &\mapsto \langle 0, 1, 0, 0, 0, 0 \rangle \\ S_2 &\mapsto \langle 0, 0, 1, 0, 0, 0 \rangle \\ S_3 &\mapsto \langle 0, 0, 0, 1, 0, 0 \rangle \\ S_4 &\mapsto \langle 0, 0, 0, 0, 1, 0 \rangle \\ S_5 &\mapsto \langle 0, 0, 0, 0, 0, 1 \rangle \end{aligned}$$

noting that we have a larger state (i.e., n bits instead of $\lceil \log_2(n) \rceil$), *but*

- transition between states is easier, *and*
- switching behaviour (and hence power consumption) is reduced.

Part 2.2: in practice, implementation (11)

Example #2: a modulo 6 ascending *or* decending counter, with cycle alert

► **Problem:** design an FSM that

1. acts as a cyclic counter modulo $n = 6$ (versus 2^n),
2. has an input d which selects between increment and decrement, and
3. has an output f which signals when a cycle occurs.

Part 2.2: in practice, implementation (11)

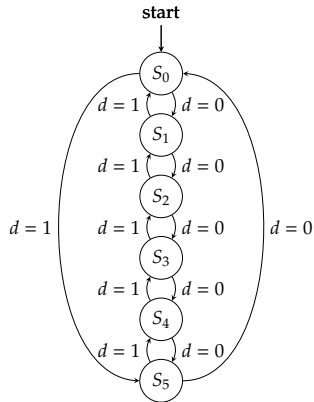
Example #2: a modulo 6 ascending *or* descending counter, with cycle alert

► Solution:

Algorithm (tabular)

Q	δ		ω		
	Q'		r	f	
	$d = 0$	$d = 1$		$d = 0$	$d = 1$
S_0	S_1	S_5	0	0	1
S_1	S_2	S_0	1	0	0
S_2	S_3	S_1	2	0	0
S_3	S_4	S_2	3	0	0
S_4	S_5	S_3	4	0	0
S_5	S_0	S_4	5	1	0

Algorithm (diagram)



Part 2.2: in practice, implementation (11)

Example #2: a modulo 6 ascending *or* decending counter, with cycle alert

► Solution:

- there are 6 abstract labels

S_0	$\mapsto$	0
S_1	$\mapsto$	1
S_2	$\mapsto$	2
S_3	$\mapsto$	3
S_4	$\mapsto$	4
S_5	$\mapsto$	5

we can represent using 6 concrete values, e.g.,

S_0	$\mapsto$	$\langle 0, 0, 0 \rangle$	$\equiv$	$000_{(2)}$
S_1	$\mapsto$	$\langle 1, 0, 0 \rangle$	$\equiv$	$001_{(2)}$
S_2	$\mapsto$	$\langle 0, 1, 0 \rangle$	$\equiv$	$010_{(2)}$
S_3	$\mapsto$	$\langle 1, 1, 0 \rangle$	$\equiv$	$011_{(2)}$
S_4	$\mapsto$	$\langle 0, 0, 1 \rangle$	$\equiv$	$100_{(2)}$
S_5	$\mapsto$	$\langle 1, 0, 1 \rangle$	$\equiv$	$101_{(2)}$

- since $2^3 = 8 > 6$, we can capture each of

1. $Q = \langle Q_0, Q_1, Q_2 \rangle \equiv$ the current state
2. $Q' = \langle Q'_0, Q'_1, Q'_2 \rangle \equiv$ the next state

in a 3-bit register (i.e., via 3 latches or flip-flops).

Part 2.2: in practice, implementation (11)

Example #2: a modulo 6 ascending *or* descending counter, with cycle alert

► Solution:

- rewriting the abstract labels yields the following concrete truth table

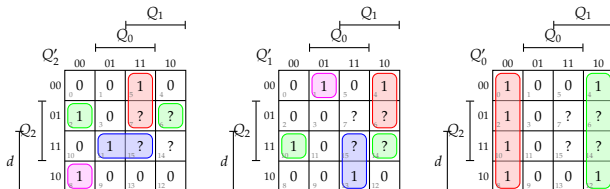
				δ			ω			
d	Q_2	Q_1	Q_0	Q'_2	Q'_1	Q'_0	r_2	r_1	r_0	f
0	0	0	0	0	0	1	0	0	0	0
0	0	0	1	0	1	0	0	0	1	0
0	0	1	0	0	1	1	0	1	0	0
0	0	1	1	1	0	0	0	1	1	0
0	1	0	0	1	0	1	1	0	0	0
0	1	0	1	0	0	0	1	0	1	1
0	1	1	0	?	?	?	?	?	?	?
0	1	1	1	?	?	?	?	?	?	?
1	0	0	0	1	0	1	0	0	0	1
1	0	0	1	0	0	0	0	0	1	0
1	0	1	0	0	0	1	0	1	0	0
1	0	1	1	0	1	0	0	1	1	0
1	1	0	0	0	1	1	1	0	0	0
1	1	0	1	1	0	0	1	0	1	0
1	1	1	0	?	?	?	?	?	?	?
1	1	1	1	?	?	?	?	?	?	?

Part 2.2: in practice, implementation (11)

Example #2: a modulo 6 ascending *or* decending counter, with cycle alert

► Solution:

- the truth table can be translated into



- doing so yields the following Boolean expressions for δ :

$$Q_2' = (\neg d \quad \wedge \quad Q_1 \quad \wedge \quad Q_0) \vee \\ (\neg d \quad \wedge \quad Q_2 \quad \wedge \quad \neg Q_0) \vee \\ (d \quad \wedge \quad Q_2 \quad \wedge \quad Q_0) \vee \\ (d \quad \wedge \quad \neg Q_2 \quad \wedge \quad \neg Q_1 \quad \wedge \quad \neg Q_0)$$

$$Q_1' = (\neg d \quad \wedge \quad \neg Q_2 \quad \wedge \quad \neg Q_1 \quad \wedge \quad Q_0) \vee \\ (\neg d \quad \wedge \quad Q_1 \quad \wedge \quad \neg Q_0) \vee \\ (d \quad \wedge \quad Q_2 \quad \wedge \quad \neg Q_0) \vee \\ (d \quad \wedge \quad Q_1 \quad \wedge \quad Q_0)$$

$$Q_0' = (\neg Q_0)$$

Part 2.2: in practice, implementation (11)

Example #2: a modulo 6 ascending *or* decending counter, with cycle alert

► Solution:

- the truth table can be translated into

		Q_1			
		Q_0			
f		00	01	11	10
00	0	0	0	0	0
01	0	1	?	?	?
11	0	0	?	?	?
10	1	0	0	0	0

d is indicated by a bracket on the left side of the table, spanning rows 01 and 11.

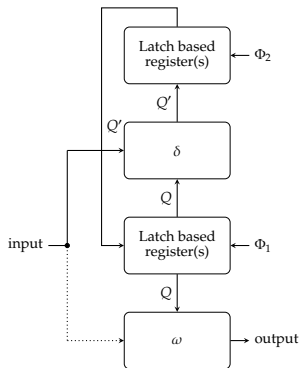
- doing so yields the following Boolean expressions for ω :

$$f = (\neg d \wedge Q_2 \wedge Q_0) \vee (d \wedge \neg Q_2 \wedge \neg Q_1 \wedge \neg Q_0)$$

Part 2.2: in practice, implementation (11)

Example #2: a modulo 6 ascending *or* decending counter, with cycle alert

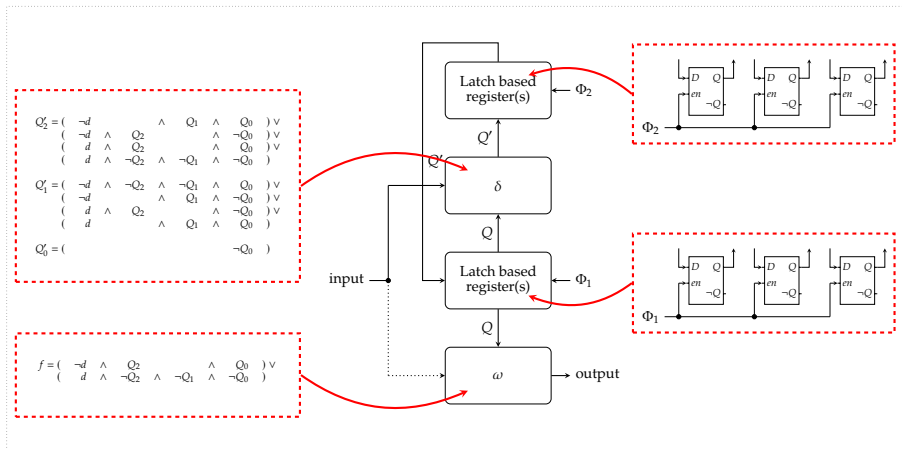
► Solution:



Part 2.2: in practice, implementation (11)

Example #2: a modulo 6 ascending *or* descending counter, with cycle alert

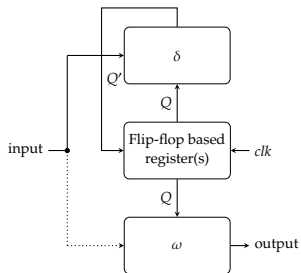
► Solution:



Part 2.2: in practice, implementation (11)

Example #2: a modulo 6 ascending *or* descending counter, with cycle alert

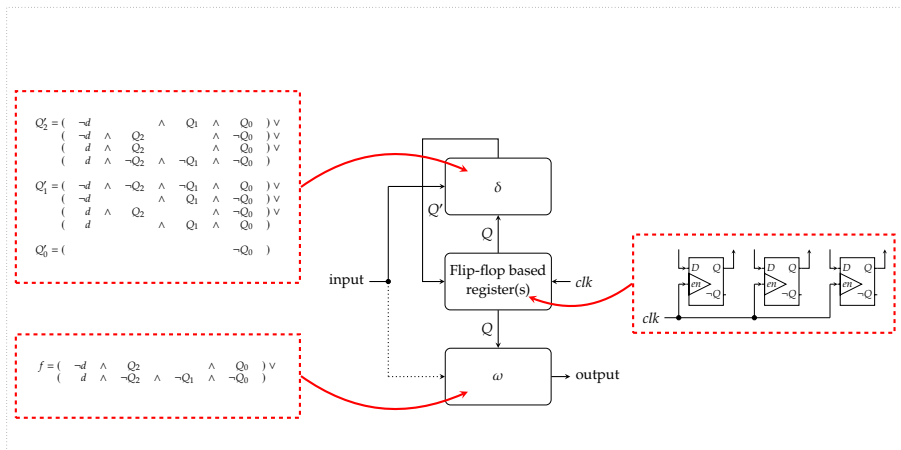
► Solution:



Part 2.2: in practice, implementation (11)

Example #2: a modulo 6 ascending *or* descending counter, with cycle alert

► Solution:



Part 2.2: in practice, implementation (12)

Example #3: a loop counter

► **Problem:** design an FSM that

1. replicates the behaviour of a controlled loop counter, e.g., `i` within a C-style `for` loop such as

```
for( int i = m; i < n; i++ ) {  
    ...  
}
```

2. has an interface that allows signalling for

the start of iteration	≡	so	$i = m$
the end of iteration	≡	when	$i = n$

focused wlog. on 4-bit values of i , m , and n .

Part 2.2: in practice, implementation (13)

Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
 - C_1 know when computation has finished (e.g., when any output $r = f(x)$ is available).
- we could implement an the interface which
 1. uses a shared clock signal to synchronise events,
 2. uses a **control protocol**, e.g., via additional *req* (or **request**) and *ack* (or **acknowledge**) signals,
 3. ...

Part 2.2: in practice, implementation (13)

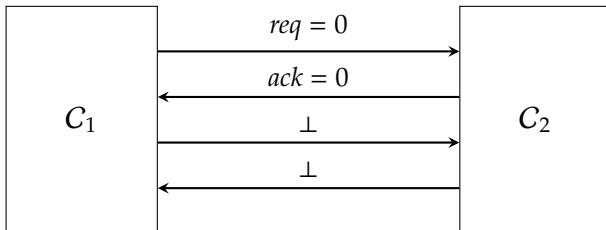
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
 - C_1 know when computation has finished (e.g., when any output $r = f(x)$ is available).
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 2. uses a **control protocol**, e.g., via additional *req* (or **request**) and *ack* (or **acknowledge**) signals,
 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

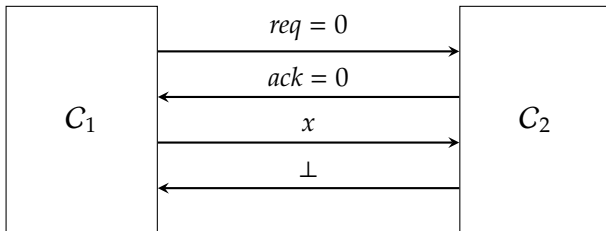
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

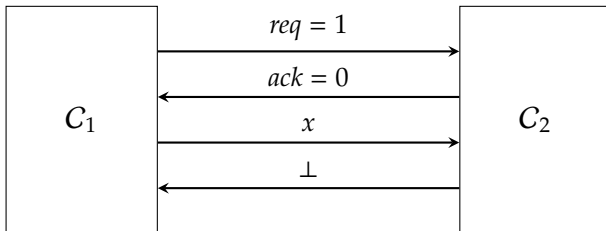
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

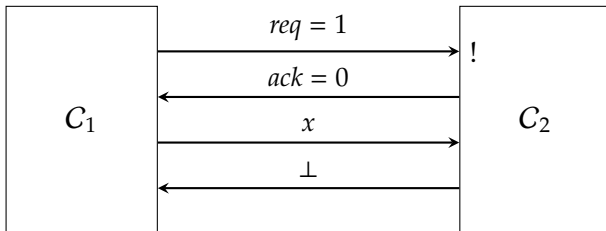
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

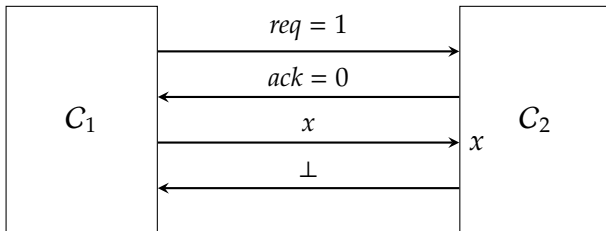
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

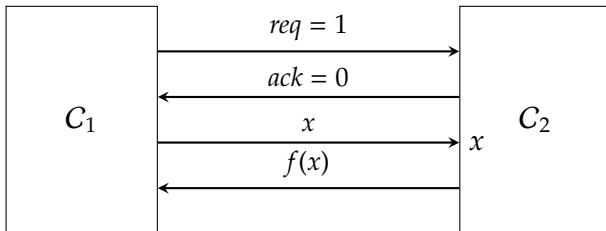
Example #3: a loop counter

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- given a user C_1 of some component C_2 , how does
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

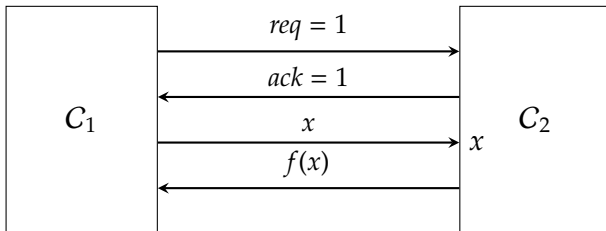
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

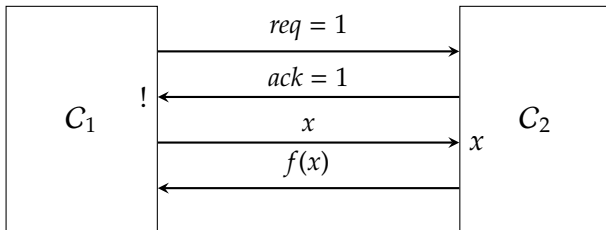
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
 - C_1 know when computation has finished (e.g., when any output $r = f(x)$ is available).
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

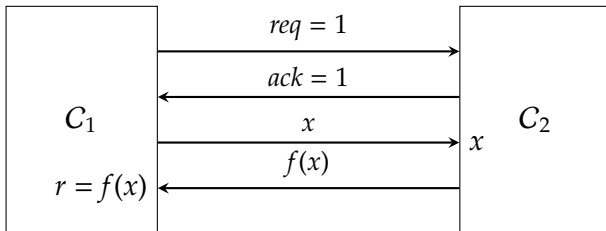
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
 - C_1 know when computation has finished (e.g., when any output $r = f(x)$ is available).
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

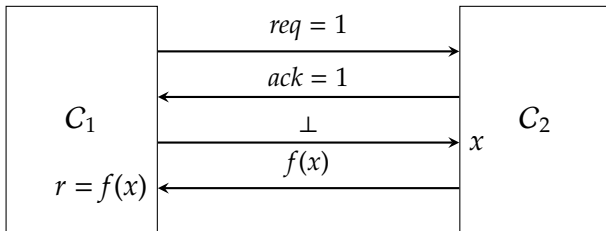
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
 - C_1 know when computation has finished (e.g., when any output $r = f(x)$ is available).
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

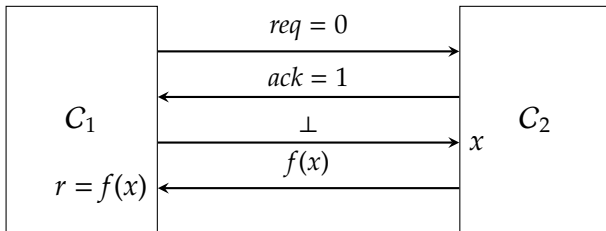
Example #3: a loop counter

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► Example:

Algorithm



Part 2.2: in practice, implementation (13)

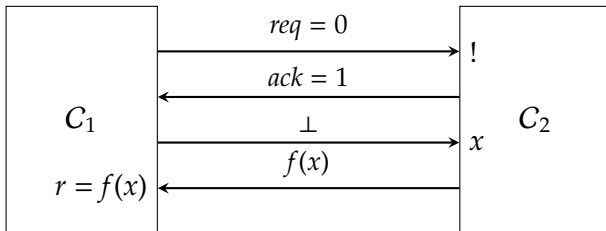
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

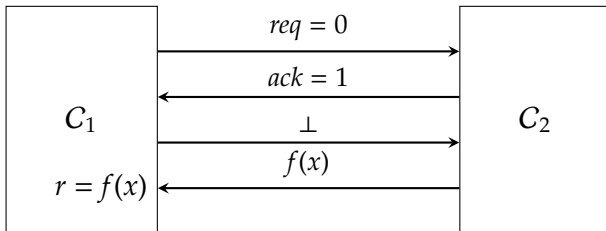
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
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 1. uses a shared clock signal to synchronise events,
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

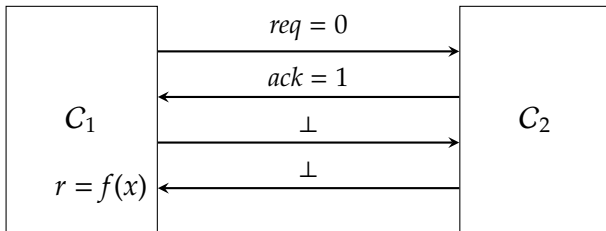
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

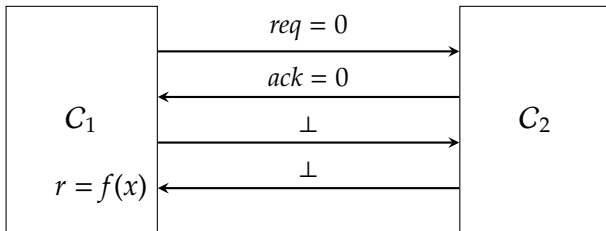
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
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- we could implement an the interface which
 1. uses a shared clock signal to synchronise events,
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 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

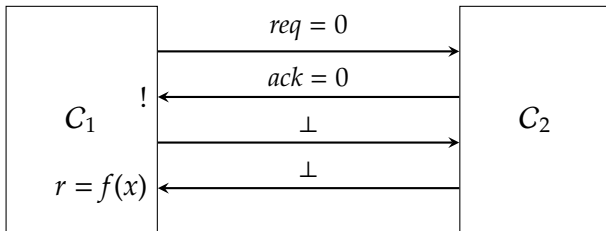
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
 - C_1 know when computation has finished (e.g., when any output $r = f(x)$ is available).
- we could implement an the interface which
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 2. uses a **control protocol**, e.g., via additional *req* (or **request**) and *ack* (or **acknowledge**) signals,
 3. ...

► Example:

Algorithm



Part 2.2: in practice, implementation (13)

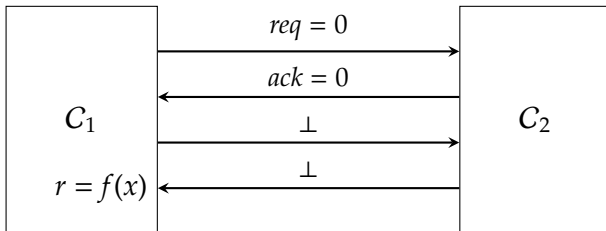
Example #3: a loop counter

► Design:

- given a user C_1 of some component C_2 , how does
 - C_2 know when to start computation (e.g., when any input x is available), and
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- we could implement an the interface which
 1. uses a shared clock signal to synchronise events,
 2. uses a **control protocol**, e.g., via additional *req* (or **request**) and *ack* (or **acknowledge**) signals,
 3. ...

► Example:

Algorithm

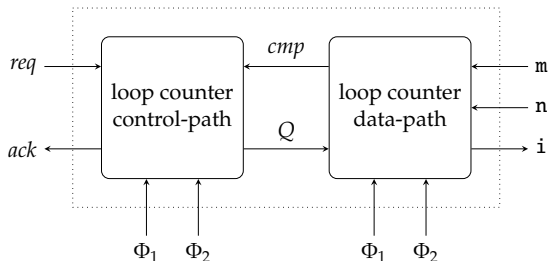


Part 2.2: in practice, implementation (14)

Example #3: a loop counter

► Design:

Circuit (latch version)



i.e., the design is *itself* the combination of

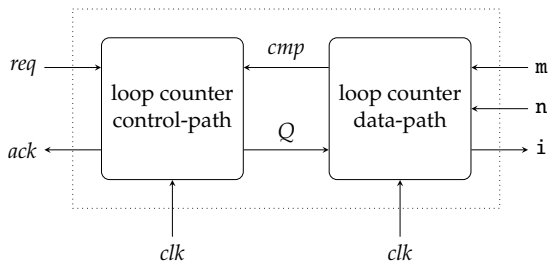
- a **data-path**, of computational and/or storage components, and
- a **control-path**, that tells components in the data-path what to do and when to do it, with the latter more overtly realised using an FSM.

Part 2.2: in practice, implementation (14)

Example #3: a loop counter

► Design:

Circuit (flip-flop version)



i.e., the design is *itself* the combination of

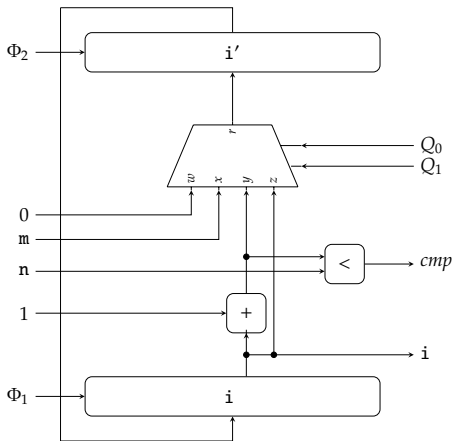
- a **data-path**, of computational and/or storage components, and
- a **control-path**, that tells components in the data-path what to do and when to do it, with the latter more overtly realised using an FSM.

Part 2.2: in practice, implementation (15)

Example #3: a loop counter

► **Solution:** the data-path.

Circuit (latch version)

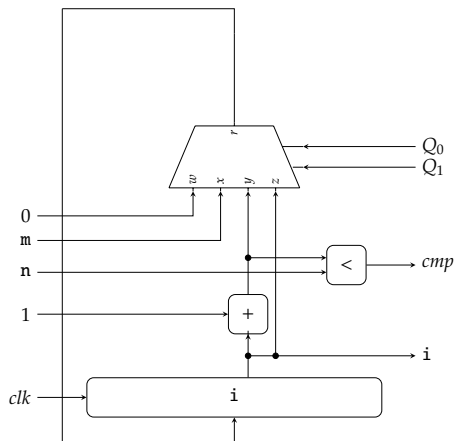


Part 2.2: in practice, implementation (15)

Example #3: a loop counter

► **Solution:** the data-path.

Circuit (flip-flop version)



Part 2.2: in practice, implementation (16)

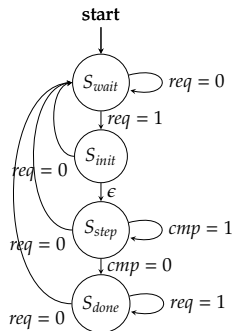
Example #3: a loop counter

► Solution: the control-path.

Algorithm (tabular)

Q	δ		ω	
	Q'	ack		
	$cmp = 0$	$cmp = 1$	$cmp = 0$	$cmp = 1$
$req = 0$	S_{wait}	S_{wait}	0	0
	S_{init}	S_{wait}	0	0
	S_{step}	S_{wait}	0	0
	S_{done}	S_{wait}	1	1
$req = 1$	S_{wait}	S_{init}	0	0
	S_{init}	S_{step}	0	0
	S_{step}	S_{step}	0	0
	S_{done}	S_{done}	1	1

Algorithm (diagram)



i.e.,

- in S_{wait} it waits for $req = 1$,
- in S_{init} it uses any input to initialise itself (e.g., setting the initial loop counter value),
- in S_{step} it performs an iteration of the loop, and
- in S_{done} it waits for $req = 0$ while setting $ack = 1$.

Part 2.2: in practice, implementation (16)

Example #3: a loop counter

► Solution: the control-path.

- there are 4 abstract labels

S_{wait}	$\mapsto$	0
S_{init}	$\mapsto$	1
S_{step}	$\mapsto$	2
S_{done}	$\mapsto$	3

we can represent using 4 concrete values, e.g.,

S_{wait}	$\mapsto$	$\langle 0, 0 \rangle$	$\equiv$	$00_{(2)}$
S_{init}	$\mapsto$	$\langle 1, 0 \rangle$	$\equiv$	$01_{(2)}$
S_{step}	$\mapsto$	$\langle 0, 1 \rangle$	$\equiv$	$10_{(2)}$
S_{done}	$\mapsto$	$\langle 1, 1 \rangle$	$\equiv$	$11_{(2)}$

- since $2^2 = 4$, we can capture each of

1. $Q = \langle Q_0, Q_1 \rangle \equiv$ the current state
2. $Q' = \langle Q'_0, Q'_1 \rangle \equiv$ the next state

in a 2-bit register (i.e., via 2 latches or flip-flops).

Part 2.2: in practice, implementation (16)

Example #3: a loop counter

► **Solution:** the control-path.

► rewriting the abstract labels yields the following concrete truth table

				δ		ω
<i>req</i>	<i>cmp</i>	Q_1	Q_0	Q'_1	Q'_0	<i>ack</i>
0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	1	0	0	0	0
0	0	1	1	0	0	1
0	1	0	0	0	0	0
0	1	0	1	0	0	0
0	1	1	0	0	0	0
0	1	1	1	0	0	1
1	0	0	0	0	1	0
1	0	0	1	1	0	0
1	0	1	0	1	1	0
1	0	1	1	1	1	1
1	1	0	0	0	1	0
1	1	0	1	1	0	0
1	1	1	0	1	0	0
1	1	1	1	1	1	1

Part 2.2: in practice, implementation (16)

Example #3: a loop counter

- **Solution:** the control-path.
 - the truth table can be translated into

		Q_1			
		Q_0	Q_1	Q_0	Q_1
Q_1'		00	01	11	10
00	0	0	0	0	0
01	0	0	0	0	0
11	0	1	1	1	1
10	0	1	1	1	1

		Q_1			
		Q_0	Q_1	Q_0	Q_1
Q_0'		00	01	11	10
00	0	0	0	0	0
01	0	0	0	0	0
11	1	0	1	0	0
10	1	0	1	1	1

- doing so yields the following Boolean expressions for δ :

$$Q_1' = (\text{req} \quad \wedge \quad Q_0) \vee (\text{req} \quad \wedge \quad Q_1)$$

$$Q_0' = (\text{req} \quad \wedge \quad \neg Q_1 \quad \wedge \quad \neg Q_0) \vee (\text{req} \quad \wedge \quad Q_1 \quad \wedge \quad Q_0) \vee (\text{req} \quad \wedge \quad \neg \text{cmp} \quad \wedge \quad Q_1)$$

Part 2.2: in practice, implementation (16)

Example #3: a loop counter

- **Solution:** the control-path.
- the truth table can be translated into

	Q_0		Q_1	
	00	01	11	10
ack	0	0	1	0
01	0	0	1	0
11	0	0	1	0
10	0	0	1	0

- doing so yields the following Boolean expressions for ω :

$$ack = Q_1 \wedge Q_0$$

Part 2.2: in practice, implementation (17)

Example #3: a loop counter

► Use-case:

- we want(ed) to implement a bit-serial multiplier, i.e.,

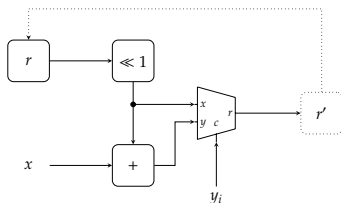
Algorithm

Input: Two unsigned, n -bit, base-2 integers x and y

Output: An unsigned, $2n$ -bit, base-2 integer
 $r = y \cdot x$

```
1  $r \leftarrow 0$ 
2 for  $i = n - 1$  downto 0 step -1 do
3    $r \leftarrow 2 \cdot r$ 
4   if  $y_i = 1$  then
5      $r \leftarrow r + x$ 
6   end
7 end
8 return  $r$ 
```

Circuit



- we *did* have the data-path,
- we *didn't* have the control-path.

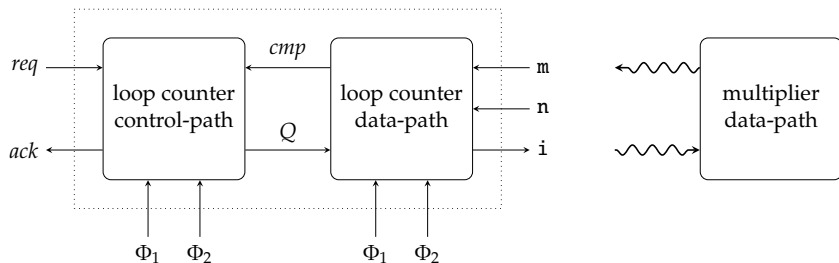
Part 2.2: in practice, implementation (18)

Example #3: a loop counter

► Use-case:

- we now have the loop counter implemented, i.e.

Circuit (latch version)



- the remaining challenge is integration, e.g., specifying
 - any additional data-path components required, and
 - how loop counter (the control-path) controls themso we end up with a bit-serial multiplier.

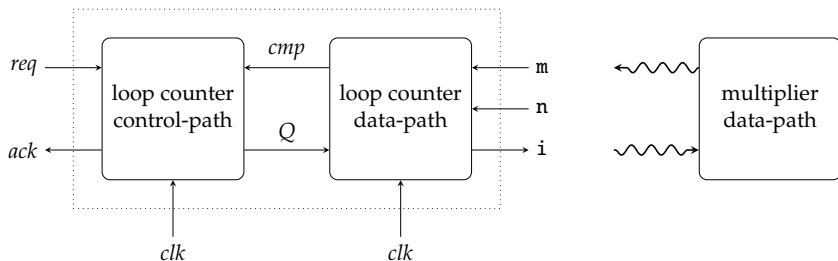
Part 2.2: in practice, implementation (18)

Example #3: a loop counter

► Use-case:

- we now have the loop counter implemented, i.e.

Circuit (flip-flop version)



- the remaining challenge is integration, e.g., specifying
 - any additional data-path components required, and
 - how loop counter (the control-path) controls themso we end up with a bit-serial multiplier.

► Take away points:

1. FSMs are abstract computational models, but we can use them to solve concrete problems, e.g.,
 - recognisers,
 - controllers,
 - ...
 - *specifications*: like an algorithm, but more easily able to cater for asynchronous events.
2. The “killer application” of FSMs for *us* is as a general-purpose way to realise controlled step-by-step forms of computation.
3. Clearly more complex problem $\Rightarrow$ more complex solution, *but*
 - same framework and process (both conceptual, *and* practical),
 - same components (e.g., interface, implementation; data-path, control-path),so difference is (arguably) creativity re. design.

Additional Reading

- ▶ *Wikipedia: Finite State Machine (FSM)*. URL: https://en.wikipedia.org/wiki/Finite-state_machine.
- ▶ D. Page. “Chapter 2: Basics of digital logic”. In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009.
- ▶ M. Sipster. “Chapter 1: Regular languages”. In: *Introduction to the Theory of Computation*. 2nd ed. Thomson Course Technology, 2006.

References

- [1] [Wikipedia: Finite State Machine \(FSM\)](https://en.wikipedia.org/wiki/Finite-state_machine). URL: https://en.wikipedia.org/wiki/Finite-state_machine (see p. 86).
- [2] D. Page. “Chapter 2: Basics of digital logic”. In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009 (see p. 86).
- [3] M. Sipster. “Chapter 1: Regular languages”. In: *Introduction to the Theory of Computation*. 2nd ed. Thomson Course Technology, 2006 (see p. 86).
- [4] M. Sipster. *Introduction to the Theory of Computation*. 2nd ed. Thomson Course Technology, 2006 (see pp. 1, 14–25).