- Agenda: **Register Machines (RMs)**, i.e.,
  1. introduce associated computational model that is "*closely related to what happens in [practical] modern digital computers*" [7, Page 199], then
  2. demonstrate how we implement said model in hardware,

  noting that

  - +ve : theoretically attractive: can be proved equivalent to a Turing machine
  - +ve : practically attractive: has clear analogies with, e.g., FSMs
  - −ve : some aspects (e.g., infinite sized registers) cannot be realised in practice
  - −ve : can be very inefficient

Definition

An **instruction** is a description of a computational step.

Definition

A **Register Machine (RM)** is specified by

- a finite number of registers, each of which can store an (infinite) natural number; $\mathsf{R}_i \in \mathbb{N}$ denotes the i-th such register for $0 \leq i < r$, and
- a program, consisting of a finite list of instructions of the form

  $$\text{label} : \text{body}$$

  such that the i-th instruction has label L_i.

Definition

An RM **configuration** is a tuple

$$C = (l, v_0, v_1, \ldots, v_{r-1})$$

where

- l is the current label, and
- v_j is the current value stored in register R_j.

Definition

A (finite or infinite) computation by some RM is captured by

$$\langle C_0, C_1, C_2, \ldots \rangle$$

i.e., a sequence of configurations such that

- for $i = 0$,

  $$C_i = (0, v_0, v_1, \ldots, v_{r-1})$$

  is the **initial configuration** where v_j is the initial value stored in register R_j,
- for $i > 0$, C_i results from applying the instruction at label L_l to

  $$C_{i-1} = (l, v_0, v_1, \ldots, v_{r-1}).$$

Definition

A finite computation by some RM is captured by

$$\langle C_0, C_1, C_2, \ldots, C_{h-1} \rangle$$

such that in the **halting configuration**

$$C_{h-1} = (l, v_0, v_1, \ldots, v_{r-1})$$

the instruction labelled L_l either

- explicitly, or intentionally forces computation to halt, i.e., is a halt instruction, or
- implicitly, or unintentionally forces computation to halt, e.g., causes an error condition.

Part 1: in theory (3)

- Example: roughly per [7, Chapter 11], consider a case where
  1. $r = 4$, i.e., it has 4 registers,
  2. each $\mathsf{R}_i \in \{0, 1, \ldots, 2^4 - 1 = 15\}$, i.e., the registers store (finite) 4-bit values,
  3. the set of available (abstract) instruction templates is

$$\mathsf{L}_i : \mathsf{R}_{addr} \leftarrow \mathsf{R}_{addr} + 1 \textbf{ then goto } \mathsf{L}_{i+1}$$
$$\mathsf{L}_i : \mathsf{R}_{addr} \leftarrow \mathsf{R}_{addr} - 1 \textbf{ then goto } \mathsf{L}_{i+1}$$
$$\mathsf{L}_i : \textbf{if } \mathsf{R}_{addr} = 0 \textbf{ then goto } \mathsf{L}_{target} \textbf{ else goto } \mathsf{L}_{i+1}$$
$$\mathsf{L}_i : \text{halt}$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1 \\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2 \\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3 \\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4 \\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5 \\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6 \\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7 \\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8 \\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9 \\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

  we can produce a **trace** of computation.

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_0 &= (0, 0, 0, 2, 0)\\
L_0 &\rightsquigarrow \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
C_1 &= (1, 0, 0, 2, 0)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_1 &= (1, 0, 0, 2, 0)\\
L_1 &\rightsquigarrow R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
C_2 &= (2, 0, 0, 1, 0)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&\mathsf{L}_0 : \textbf{if } \mathsf{R}_2 = 0 \textbf{ then goto } \mathsf{L}_5 \textbf{ else goto } \mathsf{L}_1 \\
&\mathsf{L}_1 : \mathsf{R}_2 \leftarrow \mathsf{R}_2 - 1 \textbf{ then goto } \mathsf{L}_2 \\
&\mathsf{L}_2 : \mathsf{R}_3 \leftarrow \mathsf{R}_3 + 1 \textbf{ then goto } \mathsf{L}_3 \\
&\mathsf{L}_3 : \mathsf{R}_1 \leftarrow \mathsf{R}_1 + 1 \textbf{ then goto } \mathsf{L}_4 \\
&\mathsf{L}_4 : \textbf{if } \mathsf{R}_0 = 0 \textbf{ then goto } \mathsf{L}_0 \textbf{ else goto } \mathsf{L}_5 \\
&\mathsf{L}_5 : \textbf{if } \mathsf{R}_1 = 0 \textbf{ then goto } \mathsf{L}_9 \textbf{ else goto } \mathsf{L}_6 \\
&\mathsf{L}_6 : \mathsf{R}_1 \leftarrow \mathsf{R}_1 - 1 \textbf{ then goto } \mathsf{L}_7 \\
&\mathsf{L}_7 : \mathsf{R}_2 \leftarrow \mathsf{R}_2 + 1 \textbf{ then goto } \mathsf{L}_8 \\
&\mathsf{L}_8 : \textbf{if } \mathsf{R}_0 = 0 \textbf{ then goto } \mathsf{L}_5 \textbf{ else goto } \mathsf{L}_9 \\
&\mathsf{L}_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_2 &= (2, 0, 0, 1, 0) \\
\mathsf{L}_2 &\rightsquigarrow \mathsf{R}_3 \leftarrow \mathsf{R}_3 + 1 \textbf{ then goto } \mathsf{L}_3 \\
C_3 &= (3, 0, 0, 1, 1)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&\mathsf{L}_0 : \textbf{if } \mathsf{R}_2 = 0 \textbf{ then goto } \mathsf{L}_5 \textbf{ else goto } \mathsf{L}_1 \\
&\mathsf{L}_1 : \mathsf{R}_2 \leftarrow \mathsf{R}_2 - 1 \textbf{ then goto } \mathsf{L}_2 \\
&\mathsf{L}_2 : \mathsf{R}_3 \leftarrow \mathsf{R}_3 + 1 \textbf{ then goto } \mathsf{L}_3 \\
&\mathsf{L}_3 : \mathsf{R}_1 \leftarrow \mathsf{R}_1 + 1 \textbf{ then goto } \mathsf{L}_4 \\
&\mathsf{L}_4 : \textbf{if } \mathsf{R}_0 = 0 \textbf{ then goto } \mathsf{L}_0 \textbf{ else goto } \mathsf{L}_5 \\
&\mathsf{L}_5 : \textbf{if } \mathsf{R}_1 = 0 \textbf{ then goto } \mathsf{L}_9 \textbf{ else goto } \mathsf{L}_6 \\
&\mathsf{L}_6 : \mathsf{R}_1 \leftarrow \mathsf{R}_1 - 1 \textbf{ then goto } \mathsf{L}_7 \\
&\mathsf{L}_7 : \mathsf{R}_2 \leftarrow \mathsf{R}_2 + 1 \textbf{ then goto } \mathsf{L}_8 \\
&\mathsf{L}_8 : \textbf{if } \mathsf{R}_0 = 0 \textbf{ then goto } \mathsf{L}_5 \textbf{ else goto } \mathsf{L}_9 \\
&\mathsf{L}_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_3 &= (3, 0, 0, 1, 1) \\
\mathsf{L}_3 &\rightsquigarrow \mathsf{R}_1 \leftarrow \mathsf{R}_1 + 1 \textbf{ then goto } \mathsf{L}_4 \\
C_4 &= (4, 0, 1, 1, 1)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

  we can produce a **trace** of computation:

$$
\begin{aligned}
C_4 &= (4, 0, 1, 1, 1)\\
L_4 &\rightsquigarrow \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
C_5 &= (0, 0, 1, 1, 1)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_5 &= (0, 0, 1, 1, 1)\\
L_0 &\rightsquigarrow \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
C_6 &= (1, 0, 1, 1, 1)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1 \\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2 \\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3 \\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4 \\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5 \\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6 \\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7 \\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8 \\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9 \\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_6 &= (1, 0, 1, 1, 1) \\
L_1 &\rightsquigarrow R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2 \\
C_7 &= (2, 0, 1, 0, 1)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1 \\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2 \\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3 \\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4 \\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5 \\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6 \\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7 \\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8 \\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9 \\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_7 &= (2, 0, 1, 0, 1) \\
L_2 &\rightsquigarrow R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3 \\
C_8 &= (3, 0, 1, 0, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1 \\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2 \\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3 \\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4 \\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5 \\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6 \\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7 \\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8 \\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9 \\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_8 &= (3, 0, 1, 0, 2) \\
L_3 &\rightsquigarrow R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4 \\
C_9 &= (4, 0, 2, 0, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_9 &= (4, 0, 2, 0, 2)\\
L_4 &\rightsquigarrow \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
C_{10} &= (0, 0, 2, 0, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_{10} &= (0, 0, 2, 0, 2)\\
L_0 &\rightsquigarrow \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
C_{11} &= (5, 0, 2, 0, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

  we can produce a **trace** of computation:

$$
\begin{aligned}
C_{11} &= (5, 0, 2, 0, 2)\\
L_5 &\rightsquigarrow \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
C_{12} &= (6, 0, 2, 0, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_{12} &= (6, 0, 2, 0, 2)\\
L_6 &\rightsquigarrow R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
C_{13} &= (7, 0, 1, 0, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

  we can produce a **trace** of computation:

$$
\begin{aligned}
C_{13} &= (7, 0, 1, 1, 2)\\
L_7 &\rightsquigarrow R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
C_{14} &= (8, 0, 1, 1, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1 \\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2 \\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3 \\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4 \\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5 \\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6 \\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7 \\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8 \\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9 \\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_{14} &= (8, 0, 1, 1, 2) \\
L_8 &\rightsquigarrow \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9 \\
C_{15} &= (5, 0, 1, 1, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

  we can produce a **trace** of computation:

$$
\begin{aligned}
C_{15} &= (5, 0, 1, 1, 2)\\
L_5 &\rightsquigarrow \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
C_{16} &= (6, 0, 1, 1, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1 \\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2 \\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3 \\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4 \\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5 \\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6 \\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7 \\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8 \\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9 \\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_{16} &= (6, 0, 1, 1, 2) \\
L_6 &\rightsquigarrow R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7 \\
C_{17} &= (7, 0, 0, 1, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1 \\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2 \\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3 \\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4 \\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5 \\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6 \\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7 \\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8 \\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9 \\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_{17} &= (7, 0, 0, 1, 2) \\
L_7 &\rightsquigarrow R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8 \\
C_{18} &= (8, 0, 0, 2, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

  we can produce a **trace** of computation:

$$
\begin{aligned}
C_{18} &= (8, 0, 0, 2, 2)\\
L_8 &\rightsquigarrow \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
C_{19} &= (5, 0, 0, 2, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

  we can produce a **trace** of computation:

$$
\begin{aligned}
C_{19} &= (5, 0, 0, 2, 2)\\
L_5 &\rightsquigarrow \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
C_{20} &= (9, 0, 0, 2, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&L_0 : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_1\\
&L_1 : R_2 \leftarrow R_2 - 1 \textbf{ then goto } L_2\\
&L_2 : R_3 \leftarrow R_3 + 1 \textbf{ then goto } L_3\\
&L_3 : R_1 \leftarrow R_1 + 1 \textbf{ then goto } L_4\\
&L_4 : \textbf{if } R_0 = 0 \textbf{ then goto } L_0 \textbf{ else goto } L_5\\
&L_5 : \textbf{if } R_1 = 0 \textbf{ then goto } L_9 \textbf{ else goto } L_6\\
&L_6 : R_1 \leftarrow R_1 - 1 \textbf{ then goto } L_7\\
&L_7 : R_2 \leftarrow R_2 + 1 \textbf{ then goto } L_8\\
&L_8 : \textbf{if } R_0 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_9\\
&L_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_{20} &= (9, 0, 0, 2, 2)\\
L_9 &\rightsquigarrow \text{halt}\\
C_{21} &= (9, 0, 0, 2, 2)
\end{aligned}
$$

- Example: now, given
  1. a program comprised of (concrete) instruction instances e.g.,

$$
\begin{aligned}
&\mathsf{L}_0 : \textbf{if } \mathsf{R}_2 = 0 \textbf{ then goto } \mathsf{L}_5 \textbf{ else goto } \mathsf{L}_1 \\
&\mathsf{L}_1 : \mathsf{R}_2 \leftarrow \mathsf{R}_2 - 1 \textbf{ then goto } \mathsf{L}_2 \\
&\mathsf{L}_2 : \mathsf{R}_3 \leftarrow \mathsf{R}_3 + 1 \textbf{ then goto } \mathsf{L}_3 \\
&\mathsf{L}_3 : \mathsf{R}_1 \leftarrow \mathsf{R}_1 + 1 \textbf{ then goto } \mathsf{L}_4 \\
&\mathsf{L}_4 : \textbf{if } \mathsf{R}_0 = 0 \textbf{ then goto } \mathsf{L}_0 \textbf{ else goto } \mathsf{L}_5 \\
&\mathsf{L}_5 : \textbf{if } \mathsf{R}_1 = 0 \textbf{ then goto } \mathsf{L}_9 \textbf{ else goto } \mathsf{L}_6 \\
&\mathsf{L}_6 : \mathsf{R}_1 \leftarrow \mathsf{R}_1 - 1 \textbf{ then goto } \mathsf{L}_7 \\
&\mathsf{L}_7 : \mathsf{R}_2 \leftarrow \mathsf{R}_2 + 1 \textbf{ then goto } \mathsf{L}_8 \\
&\mathsf{L}_8 : \textbf{if } \mathsf{R}_0 = 0 \textbf{ then goto } \mathsf{L}_5 \textbf{ else goto } \mathsf{L}_9 \\
&\mathsf{L}_9 : \text{halt}
\end{aligned}
$$

  2. an initial configuration, e.g.,

$$C_0 = (0, 0, 0, 2, 0)$$

we can produce a **trace** of computation:

$$
\begin{aligned}
C_{20} &= (9, 0, 0, 2, 2) \\
\mathsf{L}_9 &\rightsquigarrow \text{halt} \\
C_{21} &= (9, 0, 0, 2, 2)
\end{aligned}
$$

which demonstrates that the program copies R_2 into R_3.

Part 1: in theory (5)

- Question: is this the *only* viable example?
- Answer: many alternatives exist.

Part 1: in theory (5)

- Question: is this the *only* viable example?
- Answer: many alternatives exist, stemming, e.g., from
  1. different models, e.g.,
     - **counter machine**,
     - **Random-Access Machine (RAM)**,
     - **Random-Access Stored-Program (RASP) machine**,
     - ...

- Question: is this the *only* viable example?
- Answer: many alternatives exist, stemming, e.g., from

2. different instruction set content, e.g., a register machine with or without

$$\mathsf{L}_i : R_{addr} \leftarrow 0$$

   or "clear" instruction.

- Question: is this the *only* viable example?
- Answer: many alternatives exist, stemming, e.g., from

3. different instruction set format.

- Question: is this the *only* viable example?
- Answer: many alternatives exist, stemming, e.g., from

3. different instruction set format, e.g.,
   - **register machine** ≃ 3-operand model:
     - r registers,
     - source and destination operands can be specified independently,
     - (rough) example:

$$
\begin{array}{lcl}
 & & C_0 = (\ 0, \ 0, \ 0, \ 0, \ 0\ ) \\
\mathsf{R}_0 \leftarrow 10 & \rightsquigarrow & C_1 = (\ 1, \ 10, \ 0, \ 0, \ 0\ ) \\
\mathsf{R}_1 \leftarrow 20 & \rightsquigarrow & C_2 = (\ 2, \ 10, \ 20, \ 0, \ 0\ ) \\
\mathsf{R}_2 \leftarrow \mathsf{R}_0 + \mathsf{R}_1 & \rightsquigarrow & C_3 = (\ 3, \ 10, \ 20, \ 30, \ 0\ )
\end{array}
$$

- Question: is this the *only* viable example?
- Answer: many alternatives exist, stemming, e.g., from

3. different instruction set format, e.g.,
   - **register machine** ≃ 2-operand model:
     - r registers,
     - operands may need to be reused as source *and* destination,
     - (rough) example:

$$
\begin{array}{lcl}
 & & C_0 = (\ 0,\ 0,\ 0,\ 0,\ 0\ ) \\
\mathsf{R}_0 \leftarrow 10 & \rightsquigarrow & C_1 = (\ 1,\ 10,\ 0,\ 0,\ 0\ ) \\
\mathsf{R}_1 \leftarrow 20 & \rightsquigarrow & C_2 = (\ 2,\ 10,\ 20,\ 0,\ 0\ ) \\
\mathsf{R}_0 \leftarrow \mathsf{R}_0 + \mathsf{R}_1 & \rightsquigarrow & C_3 = (\ 3,\ 30,\ 20,\ 0,\ 0\ )
\end{array}
$$

- Question: is this the *only* viable example?
- Answer: many alternatives exist, stemming, e.g., from

3. different instruction set format, e.g.,
   - **accumulator machine** $\simeq$ 1-operand model:
     - may have $r > 1$ register, but there is 1 special-purpose case termed the **accumulator**,
     - operations implicitly use accumulator for source and/or destination operands,
     - (rough) example:

$$
\begin{array}{lcl}
 & \rightsquigarrow & C_0 = (\ 0,\ 0,\ 0,\ 0,\ 0\ ) \\
\mathsf{A} \leftarrow 10 & \rightsquigarrow & C_1 = (\ 1,\ 10,\ 0,\ 0,\ 0\ ) \\
\mathsf{A} \leftarrow \mathsf{A} + 20 & \rightsquigarrow & C_2 = (\ 2,\ 30,\ 0,\ 0,\ 0\ )
\end{array}
$$

- Question: is this the *only* viable example?
- Answer: many alternatives exist, stemming, e.g., from

3. different instruction set format, e.g.,
   - **stack machine** $\simeq$ 0-operand model:
     - may have $r > 1$ register, but managed per a **stack** (i.e., FILO-style) policy,
     - operations implicitly use stack for source and/or destination operands,
     - (rough) example:

$$
\begin{array}{lcl}
 & \rightsquigarrow & C_0 = (\ 0,\ 0,\ 0,\ 0,\ 0\ ) \\
\text{push } 20 & \rightsquigarrow & C_1 = (\ 1,\ 20,\ 0,\ 0,\ 0\ ) \\
\text{push } 10 & \rightsquigarrow & C_2 = (\ 2,\ 10,\ 20,\ 0,\ 0\ ) \\
\text{add} & \rightsquigarrow & C_3 = (\ 3,\ 30,\ 0,\ 0,\ 0\ ) \\
\text{pop} & \rightsquigarrow & C_4 = (\ 4,\ 0,\ 0,\ 0,\ 0\ )
\end{array}
$$

Definition

Consider a sequence

$$x = \langle x_0, x_1, \ldots, x_{n-1} \rangle$$

where, for each $0 \leq i < n$ we have $x_i \in \mathbb{N}$. The associated **Gödel encoding** (or **Gödel numbering**) is

$$\hat{x} = \prod_{i=0}^{i<n} p_i^{x_i} = p_0^{x_0} \cdot p_1^{x_1} \cdots p_{n-1}^{x_{n-1}}$$

where p_i is the i-th prime, i.e., $p_0 = 2$, $p_1 = 3$, $p_2 = 5$, and so on. Due to Euclid's unique prime-factorisation theorem, factoring $\hat{x}$ allows recovery of x.

$\therefore$ we can represent *anything* using elements of $\mathbb{N}$, e.g., per [8, Section VII.A],

- let 6 represent "0",
- let 5 represent "=", then
- the logical statement "0 = 0" can be represented as

$$2^6 \cdot 3^5 \cdot 5^6 = 243,000,000.$$

- Concept:
  - One can view i.e.,

$$\begin{aligned} \text{instruction} &= \text{information} \mapsto \text{what to do} \\ \text{data} &= \text{information} \mapsto \text{what to do it on/with} \end{aligned}$$

and Gödel encoding allows numerical representation of *either* form of information, e.g.,

$$\begin{aligned} 1 &\mapsto \text{“compute an addition” if it represents an instruction} \\ 1 &\mapsto \text{“the integer one” if it represents some data} \end{aligned}$$

are different, valid interpretations of the same number.

- Concept:

- One can view

  (human-readable) instruction $\simeq$ abstraction of (machine-readable) control information.

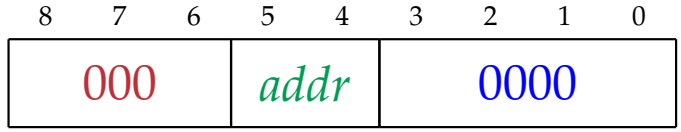
- Concept:

- These facts suggest a strategy: an RM is an FSM in disguise, in the sense that
  1. an RM configuration is an FSM state, and
  2. the RM program determines the FSM transition function,

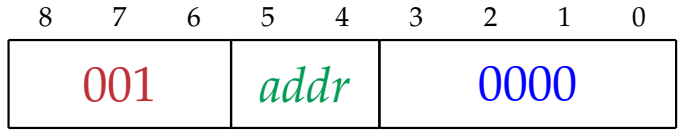
  so we could therefore
  - use, e.g., Gödel encoding or variant thereof, to encode instructions into numerical **machine code**,
  - store the machine code in memory,
  - have our implementation decode machine code into appropriate control signals.

- Design: specify an instruction encoding, e.g.,

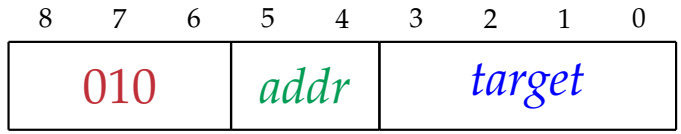
$L_i : R_{addr} \leftarrow R_{addr} + 1$ **then goto** L_{i+1} $\mapsto$



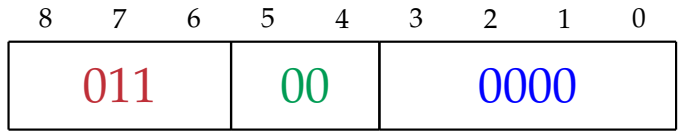
$L_i : R_{addr} \leftarrow R_{addr} - 1$ **then goto** L_{i+1} $\mapsto$



L_i : **if** $R_{addr} = 0$ **then goto** L_{target} **else goto** L_{i+1} $\mapsto$



L_i : halt $\mapsto$



such that

$$L_i : \textbf{if } R_2 = 0 \textbf{ then goto } L_5 \textbf{ else goto } L_{i+1} \mapsto 010100101_{(2)}.$$

- Observation(s):
  - we opted to encode instructions as 9-element sequences of bits, noting that

$$000010000_{(2)} \equiv 010_{(16)} \equiv 16_{(10)}$$

and that

$$000010000_{(2)} \mapsto \text{“increment register one” if it represents an instruction}$$
$$000010000_{(2)} \mapsto \text{“the integer sixteen” if it represents some data}$$

- Observation(s):
  - we opted for a means of encoding that could be described as

$$\begin{array}{rcccccc} \hat{x} & = & x_0 & \| & x_1 & \| & x_2 \\ & \equiv & x_0 \cdot 2^0 & + & x_1 \cdot 2^4 & + & x_2 \cdot 2^6 \end{array}$$

    so that decoding means extraction of contiguous bits from $\hat{x}$: i.e.,

$$\begin{array}{rcccc} x_0 & = & \hat{x}_{3\ldots0} & \equiv & (\hat{x} \gg 0) \wedge F_{(16)} \\ x_1 & = & \hat{x}_{5\ldots4} & \equiv & (\hat{x} \gg 4) \wedge 3_{(16)} \\ x_2 & = & \hat{x}_{8\ldots6} & \equiv & (\hat{x} \gg 6) \wedge 7_{(16)} \end{array}$$

  - so this is *similar* to Gödel encoding, but *much* more efficient.

- Observation(s):
  - this design clearly isn't the *only* one possible:
    - some aspects stem from *constraints*:
      - with $2^2 = 4$ registers, we need at least a 2-bit green field,
      - ...
    - some aspects stem from *choices*:
      - a 4-bit blue field limits our programs to $2^4 = 16$ instructions,
      - the assignment of values that identify instruction types is somewhat arbitrary,
      - the order and location of fields within the bit-sequence is somewhat arbitrary,
      - ...

- Design: encode our original program

$$
\begin{array}{lcll}
\mathsf{L}_0 : \textbf{if } \mathsf{R}_2 = 0 \textbf{ then goto } \mathsf{L}_5 \textbf{ else goto } \mathsf{L}_1 & \mapsto & 010100101_{(2)} & = 0A5_{(16)} \\
\mathsf{L}_1 : \mathsf{R}_2 \leftarrow \mathsf{R}_2 - 1 \textbf{ then goto } \mathsf{L}_2 & \mapsto & 001100000_{(2)} & = 060_{(16)} \\
\mathsf{L}_2 : \mathsf{R}_3 \leftarrow \mathsf{R}_3 + 1 \textbf{ then goto } \mathsf{L}_3 & \mapsto & 000110000_{(2)} & = 030_{(16)} \\
\mathsf{L}_3 : \mathsf{R}_1 \leftarrow \mathsf{R}_1 + 1 \textbf{ then goto } \mathsf{L}_4 & \mapsto & 000010000_{(2)} & = 010_{(16)} \\
\mathsf{L}_4 : \textbf{if } \mathsf{R}_0 = 0 \textbf{ then goto } \mathsf{L}_0 \textbf{ else goto } \mathsf{L}_5 & \mapsto & 010000000_{(2)} & = 080_{(16)} \\
\mathsf{L}_5 : \textbf{if } \mathsf{R}_1 = 0 \textbf{ then goto } \mathsf{L}_9 \textbf{ else goto } \mathsf{L}_6 & \mapsto & 010011001_{(2)} & = 099_{(16)} \\
\mathsf{L}_6 : \mathsf{R}_1 \leftarrow \mathsf{R}_1 - 1 \textbf{ then goto } \mathsf{L}_7 & \mapsto & 001010000_{(2)} & = 050_{(16)} \\
\mathsf{L}_7 : \mathsf{R}_2 \leftarrow \mathsf{R}_2 + 1 \textbf{ then goto } \mathsf{L}_8 & \mapsto & 000100000_{(2)} & = 020_{(16)} \\
\mathsf{L}_8 : \textbf{if } \mathsf{R}_0 = 0 \textbf{ then goto } \mathsf{L}_5 \textbf{ else goto } \mathsf{L}_9 & \mapsto & 010000101_{(2)} & = 085_{(16)} \\
\mathsf{L}_9 : \text{halt} & \mapsto & 011000000_{(2)} & = 0C0_{(16)}
\end{array}
$$

such that

- we use

$$\mathsf{MEM} = \langle 0A5_{(16)}, 060_{(16)}, \ldots, 0C0_{(16)} \rangle,$$

  a 10-element memory,
- each $\mathsf{MEM}[i]$ is a 9-bit encoding of the instruction labelled L_i.

Definition

The **stored program** concept implies that a mutable program is stored in some form of memory. A **stored program computer** is said to **execute** a stored program, and so the instructions within it: it does so by loading (or fetching) those instructions, before decoding and executing them.

Definition

The **Program Counter (PC)** (or **instruction pointer**) is a special-purpose register that holds the address of the next instruction to be executed.

Definition

The **Instruction Register (IR)** is a special-purpose register that holds the instruction currently being executed.

Definition

An **instruction decoder** is typically a combinatorial component, either way tasked with taking an encoded instruction as input and producing a set of associated control signals as output; said control signals are used by the rest of the implementation to control execution of the instruction.

Definition

The **fetch-decode-execute cycle** (aka. **instruction cycle**) is a 3-stage process

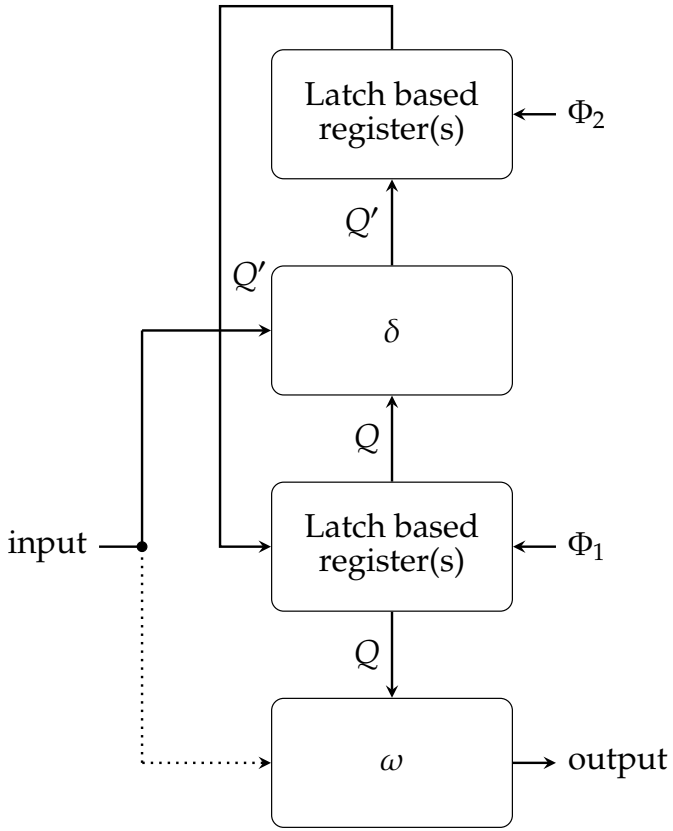
1. fetch stage :
   - 1.a. load instruction into IR $\mapsto$ IR $\leftarrow$ MEM[PC]
   - 1.b. increment PC $\mapsto$ PC $\leftarrow$ PC + 1
2. decode stage : *decide* what instruction in IR *means*, i.e., translate IR into control signals which reflect instruction semantics
3. execute stage : *do* whatever instruction in IR *means*, i.e., apply instruction semantics

which describes execution of instructions; in some cases it makes sense to consider a 5-stage process by adding

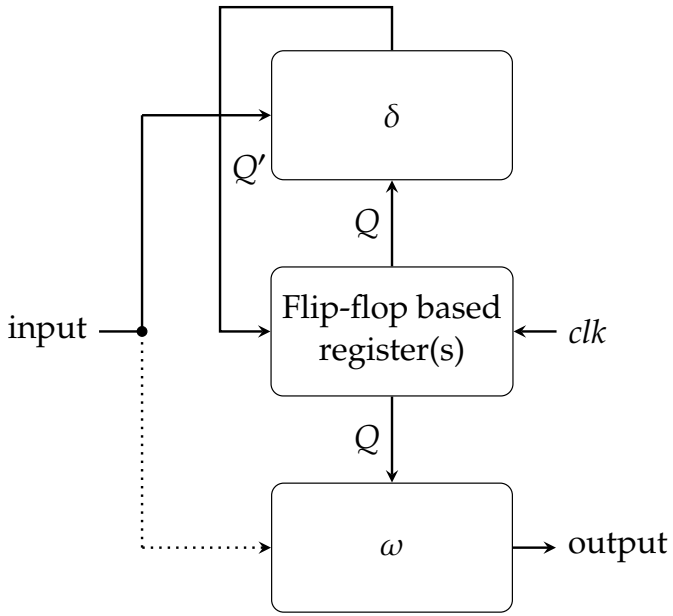
4. memory access stage : perform any memory accesses (e.g., loads or stores) required
5. write-back (or commit) stage : store result(s) stemming from instruction execution (e.g., computation)

i.e., expanding the execute stage to be more precise.

Recall

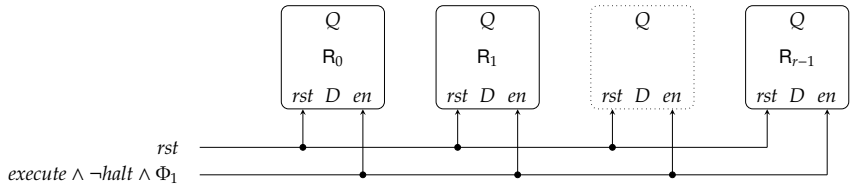

Latch based register(s)
Φ_2
Q'
Q'
δ
Q
input
Latch based register(s)
Φ_1
Q
ω
output

Recall

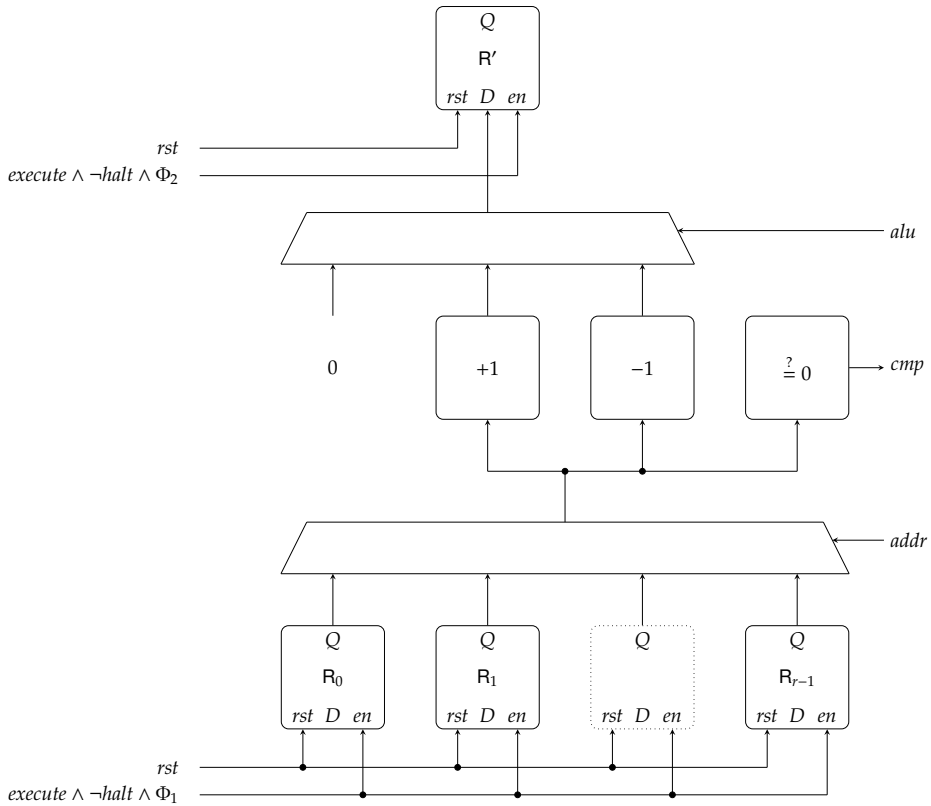


Circuit

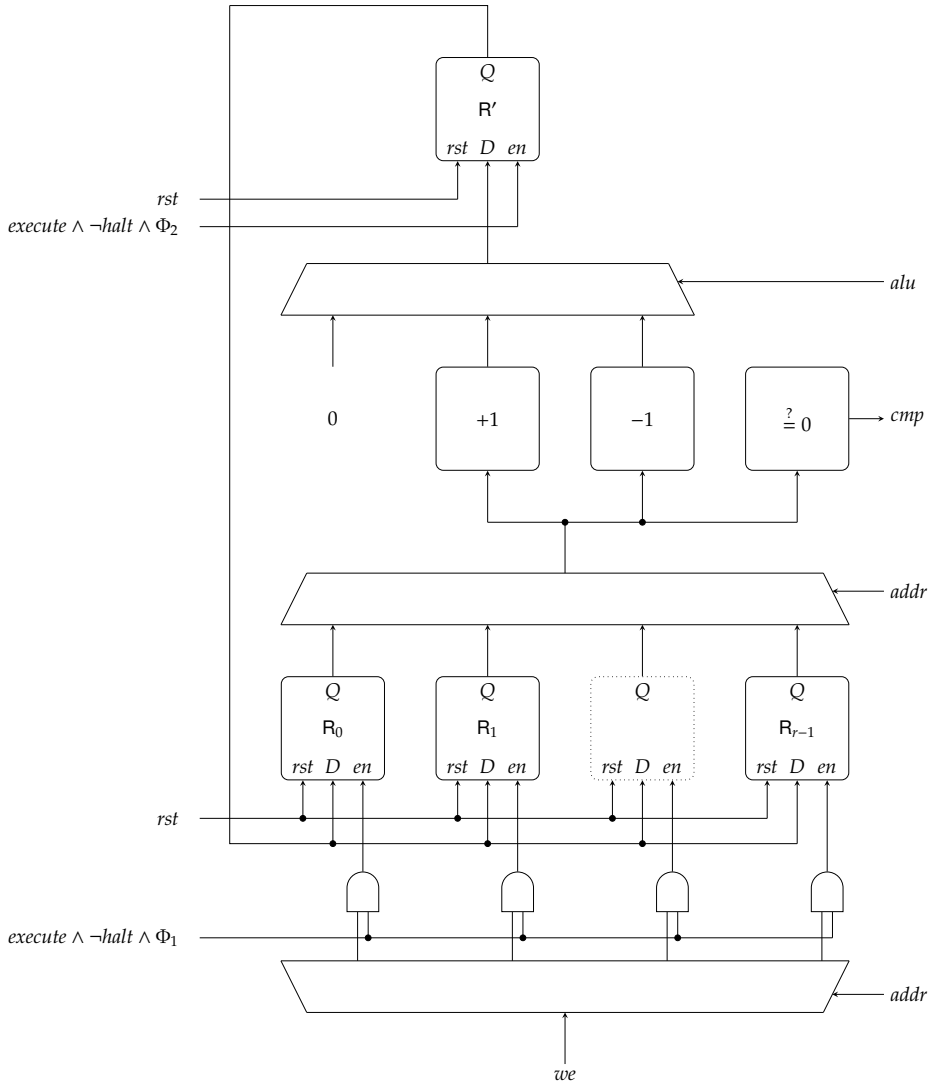

Q
R′
rst D en
rst
execute ∧ ¬halt ∧ Φ2


Q
R0
rst D en
Q
R1
rst D en
Q
rst D en
Q
Rr−1
rst D en
rst
execute ∧ ¬halt ∧ Φ1

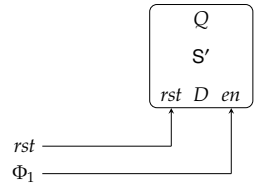
Circuit

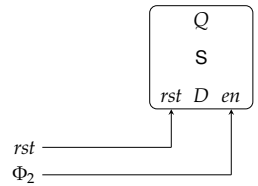

Q
R′
rst D en
rst
execute ∧ ¬halt ∧ Φ2
alu
0
+1
−1
?= 0
cmp
addr
Q
R0
rst D en
Q
R1
rst D en
Q
rst D en
Q
Rr−1
rst D en
rst
execute ∧ ¬halt ∧ Φ1

Circuit

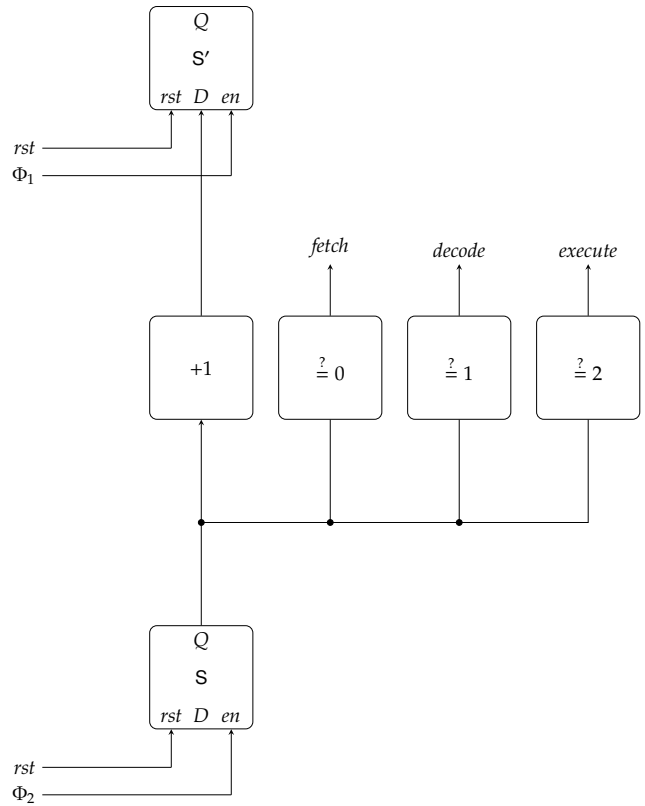

Q
R′
rst D en
rst
execute ∧ ¬halt ∧ Φ2
alu
0
+1
−1
?= 0
cmp
addr
Q
R0
rst D en
Q
R1
rst D en
Q
rst D en
Q
Rr−1
rst D en
rst
execute ∧ ¬halt ∧ Φ1
addr
we

Circuit

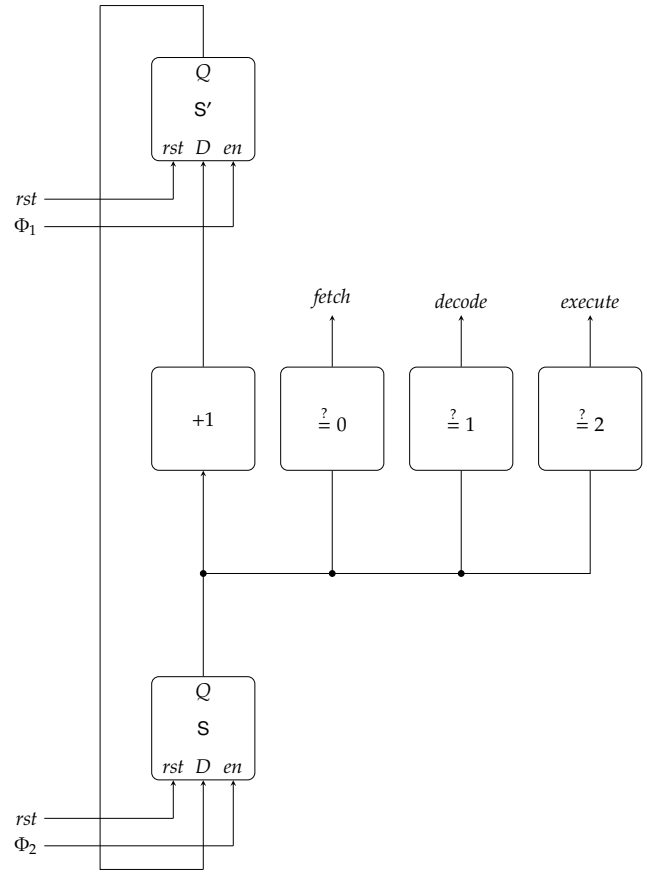




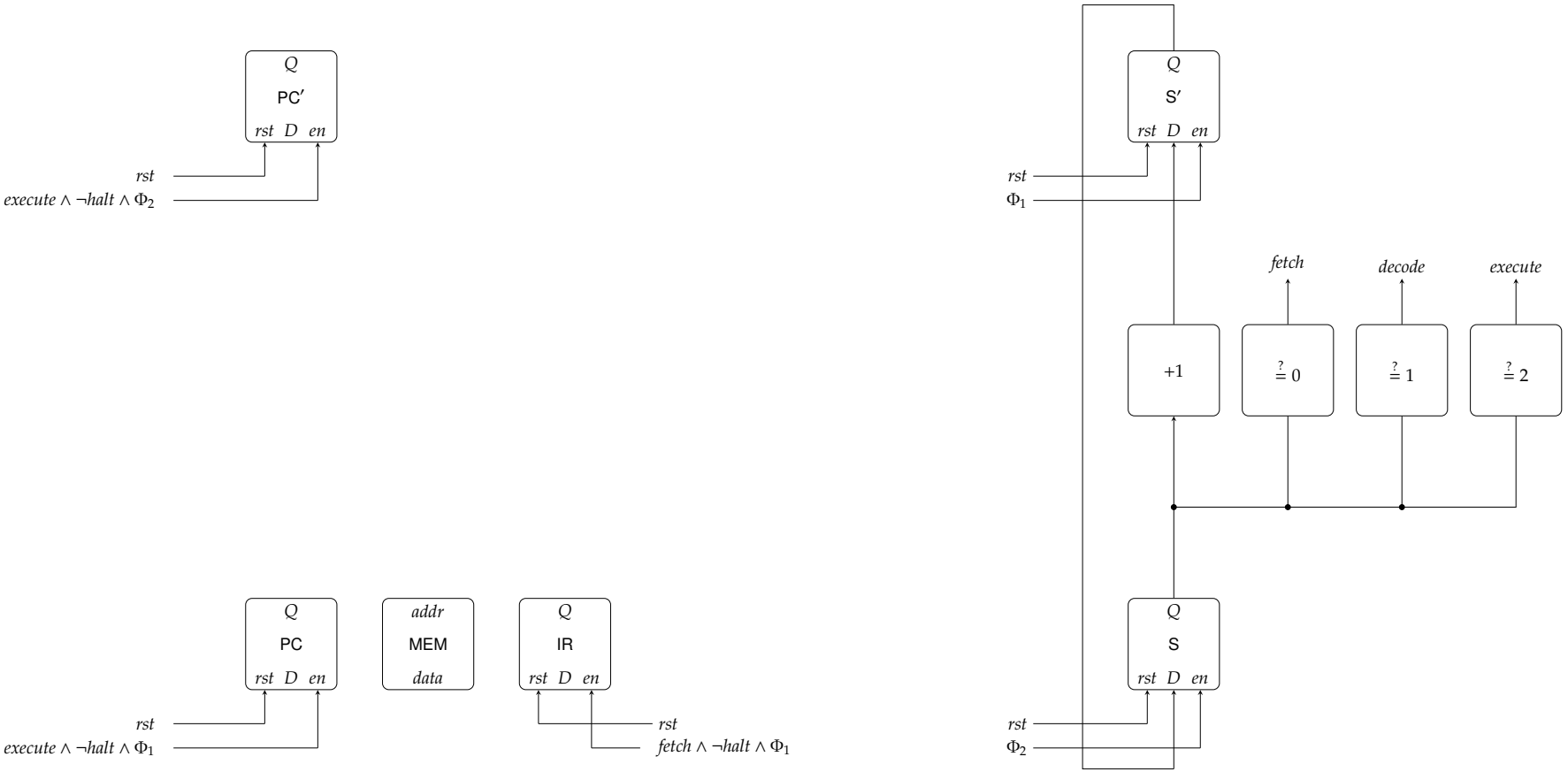
Circuit



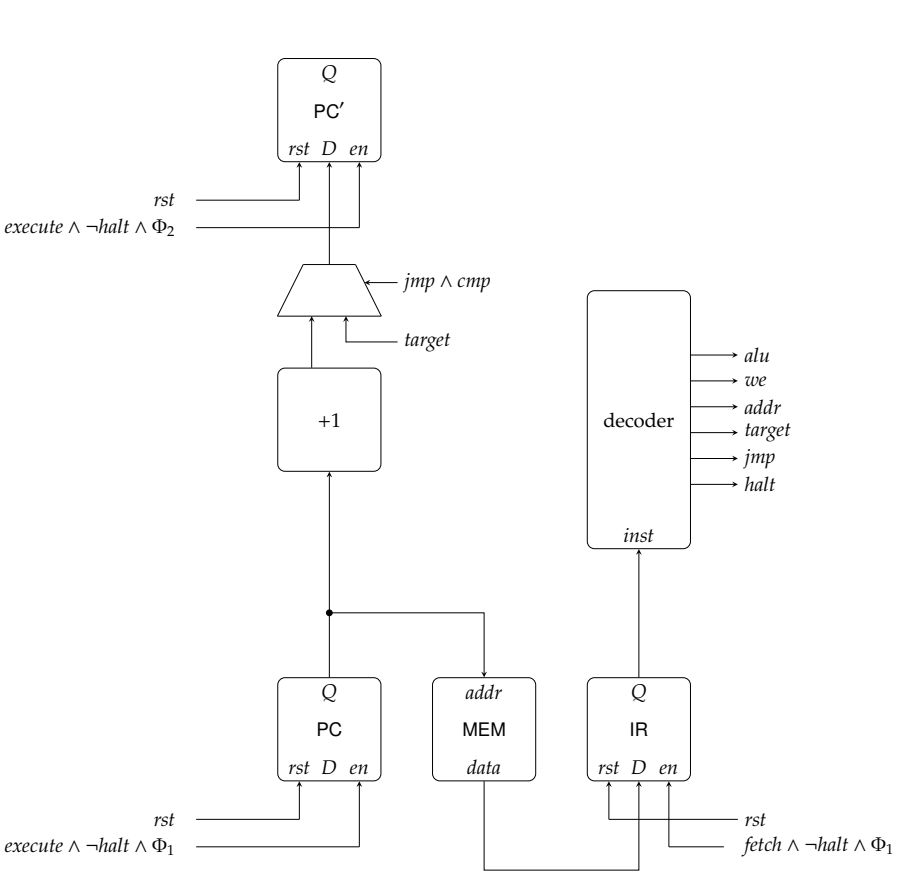
Circuit

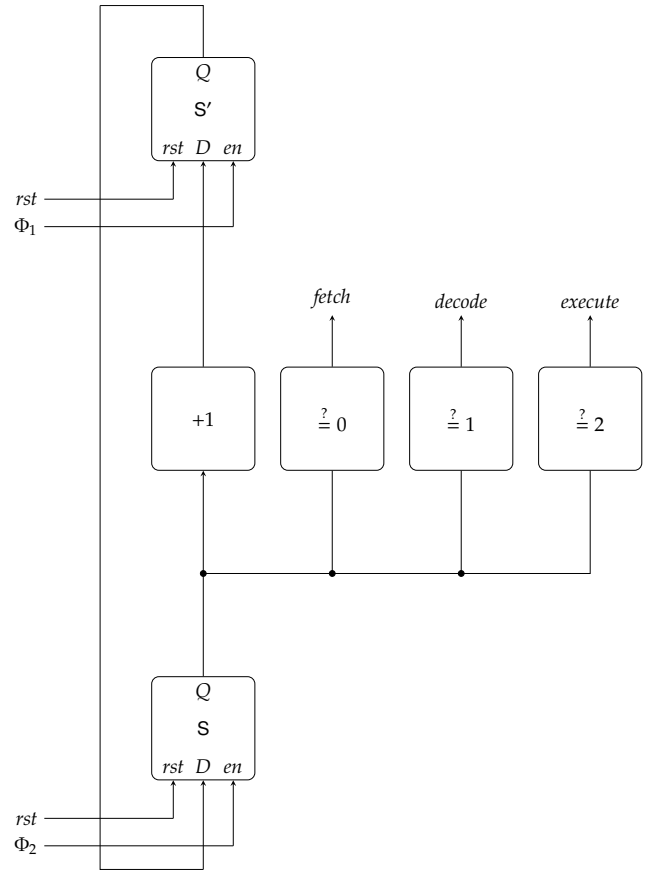


Circuit

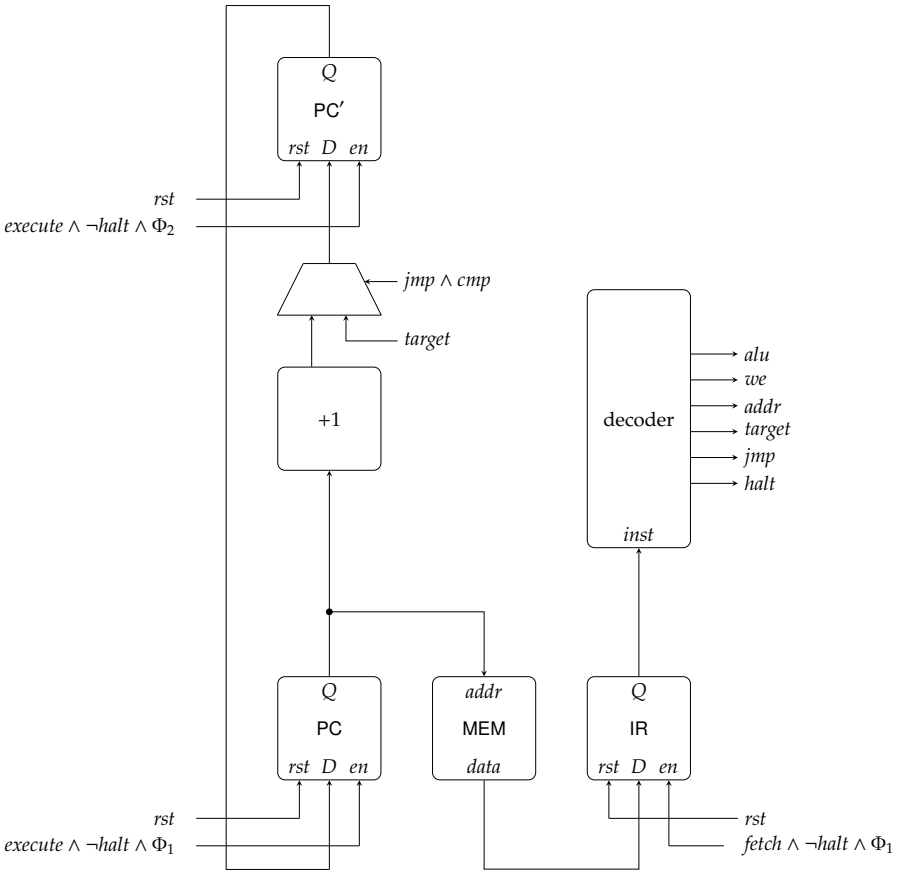

Q
PC′
rst D en
rst
execute ∧ ¬halt ∧ Φ2
Q
S′
rst D en
rst
Φ1
fetch
decode
execute
+1
? = 0
? = 1
? = 2
Q
PC
rst D en
rst
execute ∧ ¬halt ∧ Φ1
addr
MEM
data
Q
IR
rst D en
rst
fetch ∧ ¬halt ∧ Φ1
Q
S
rst D en
rst
Φ2

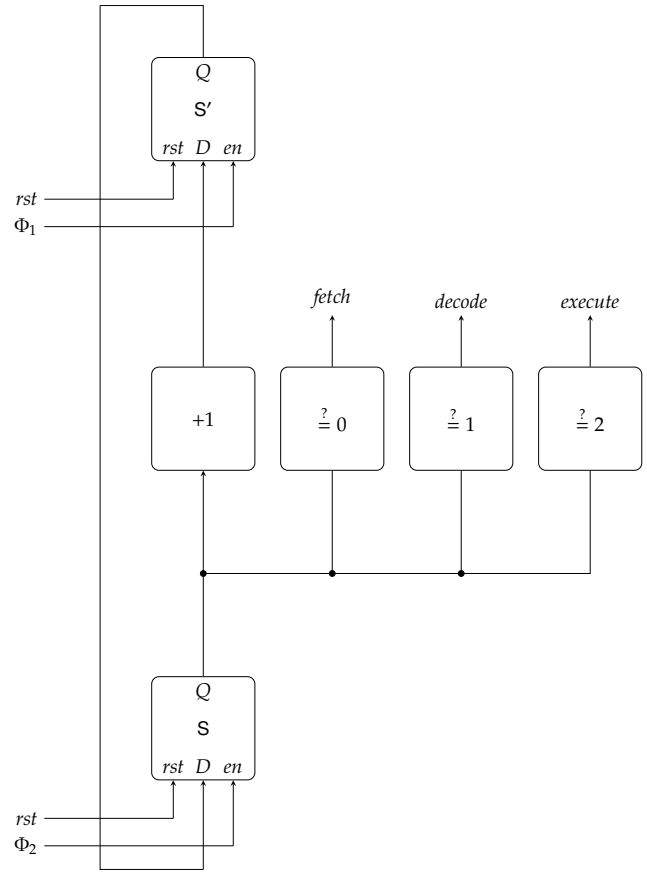
Circuit


Q
PC′
rst D en
rst
execute ∧ ¬halt ∧ Φ2
jmp ∧ cmp
target
+1
decoder
alu
we
addr
target
jmp
halt
inst
Q
PC
rst D en
addr
MEM
data
Q
IR
rst D en
rst
execute ∧ ¬halt ∧ Φ1
rst
fetch ∧ ¬halt ∧ Φ1


Q
S′
rst D en
rst
Φ1
fetch
decode
execute
+1
? = 0
? = 1
? = 2
Q
S
rst D en
rst
Φ2

Circuit


Q
PC′
rst D en
rst
execute ∧ ¬halt ∧ Φ2
jmp ∧ cmp
target
+1
decoder
alu
we
addr
target
jmp
halt
inst
Q
PC
rst D en
addr
MEM
data
Q
IR
rst D en
rst
execute ∧ ¬halt ∧ Φ1
rst
fetch ∧ ¬halt ∧ Φ1


Q
S′
rst D en
rst
Φ1
fetch
decode
execute
+1
? = 0
? = 1
? = 2
Q
S
rst D en
rst
Φ2

Conclusions

- Take away points:
  1. A central aim here is to demonstrate that, per

$$\begin{aligned}
\text{combinatorial logic} &\rightsquigarrow \text{fixed function, not stateful} \\
\text{sequential logic} &\rightsquigarrow \text{fixed function, stateful} \\
\text{FSM} &\rightsquigarrow \text{fixed function, stateful} \\
\text{RM} &\rightsquigarrow \text{not fixed function, stateful} \\
&\vdots \\
\text{micro-processor} &\rightsquigarrow \text{not fixed function, stateful}
\end{aligned}$$

     and although our register machine is *still* limited,
     - it has started to exhibit the characteristics of a *real* micro-processor, *and*
     - we can *still* reason end-to-end about the implementation.
  2. In doing so we've encountered some fundamental concepts, e.g.,
     - instruction encoding and decoding,
     - the fetch-decode-execute cycle,
     - the role of PC and IR in supporting it,
     - ...

     which we'll revisit and refine.

Additional Reading

- *Wikipedia: Register machine*. URL: https://en.wikipedia.org/wiki/Register_machine.
- *Wikipedia: Counter machine*. URL: https://en.wikipedia.org/wiki/Counter_machine.
- *Wikipedia: Random-access machine*. URL: https://en.wikipedia.org/wiki/Random-access_machine.
- *Wikipedia: Random-access stored-program machine*. URL: https://en.wikipedia.org/wiki/Random-access_stored-program_machine.
- *Wikipedia: Gödel numbering*. URL: https://en.wikipedia.org/wiki/G%C3%B6del_numbering.
- *Wikipedia: Machine code*. URL: https://en.wikipedia.org/wiki/Machine_code.

References

[1] *Wikipedia: Counter machine*. URL: https://en.wikipedia.org/wiki/Counter_machine (see p. 60).

[2] *Wikipedia: Gödel numbering*. URL: https://en.wikipedia.org/wiki/G%C3%B6del_numbering (see p. 60).

[3] *Wikipedia: Machine code*. URL: https://en.wikipedia.org/wiki/Machine_code (see p. 60).

[4] *Wikipedia: Random-access machine*. URL: https://en.wikipedia.org/wiki/Random-access_machine (see p. 60).

[5] *Wikipedia: Random-access stored-program machine*. URL: https://en.wikipedia.org/wiki/Random-access_stored-program_machine (see p. 60).

[6] *Wikipedia: Register machine*. URL: https://en.wikipedia.org/wiki/Register_machine (see p. 60).

[7] M. Minsky. *Computation: Finite and Infinite Machines*. Prentice Hall, 1967 (see pp. 1, 4).

[8] E. Nagel and J.R. Newman. *Gödel's Proof*. Routledge, 1958 (see p. 36).